DYNAMICAL SYMMETRIES IN THE DIRAC EQUATION (ON THE NATURE OF "HIDDEN SYMMETRY" OR "ACCIDENTAL DEGENERACY" IN THE KEPLER PROBLEM)

Tamar Khachidze and Anzor Khelashvili,
(Tbilisi State University, Tbilisi, Georgia)

The aim of our talk is to determine what the Dirac equation tells us about this problem. We can see that the very interesting physical picture takes place.

For the Kepler problem in the Dirac equation Johnson and Lippmann (JL) published very brief abstract in Physical Review at 1950. They had written that there is an additional conserved quantity

$$A=\vec{\sigma}\cdot\vec{r}r^{-1}-i\left(\frac{\hbar c}{e^2}\right)\left(mc^2\right)^{-1}j\rho_1\left(H-mc^2\rho_3\right), \qquad \textbf{(1)}$$

which plays the same role in Dirac equation, as the LRL vector in Schroedinger equation.
As regard of derivation of this operator to our surprise was not published anywhere in scienific literature up to nowadays. (one of the curious fact in the history of physics of 20th century), and the demonstration of its commutativity with the Dirac Hamiltonian is considered as a tadious task.

Recently, we developed rather simple and transparent way for deriving the JL operator. We obtained this operator and at the same time proved its commutativity with the Hamiltonian.

After we consider necessity to be convinced, that the Coulomb problem is distinguishable in this point of view. We considered the Dirac equation in arbitrary central potential, $V(r)$ **and shown that the symmetry, which will be defined more precisely below, takes place only for Coulomb potential.**

We have choosen the subject of todays' talk so that it is consonant with 1966 School, when several lectures were devoted to the problem of hidden symmetry – lectures by Alilluev, Matveenko, Popov and Werle. During several decades these lectures were the only source, from which the young students and scientists could learn about this topic and the Proceedings of that School is excellent monography in many subjects of Theoretical Physics.

In some respect my todays' lecture is a continuation (but not analytic) of those old lectures.

First of all consider the Dirac Hamiltonian in central field, $V(r)$**, which is the 4th component of the Lorentz vector:**

$$H = \vec{\alpha}\cdot\vec{p} + \beta m + V(r) \qquad (2)$$

The so-called Dirac's K-operator, defined as

$$K = \beta\left(\vec{\Sigma}\cdot\vec{l} + 1\right) \qquad (3)$$

commutes with this Hamiltonian for arbitrary $V(r)$**,** $\left[K, H\right] = 0$

Here $\vec{\Sigma}$ **is the spin matrix**

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

and $\vec{l}$ **- orbital momentum vector-operator.**

It is evident that the eigenstates of Hamiltonian also are labeled by eigenvalues of K**. For example, the well-known Sommerfeld formula for the Coulombic spectrum looks like**

$$E_{n,|\kappa|} = m\left[1+\frac{(Z\alpha)^2}{\left(n-|\kappa|+\sqrt{\kappa^2-(Z\alpha)^2}\right)^2}\right]^{-1/2}$$

It manifests explicit dependence on κ, more precisely on $|\kappa| = j+1/2$.

It is surprising that for other solvable potentials the degeneracy with respect to signs of κ does not take place.

Is it peculiar only for Coulomb potential?

Let us consider Hamiltonian (2) with arbitrary potential and require the degeneracy in $\pm\kappa$. It is natural that for description of this degeneracy one has to find an operator, that mixes these two signs. It is clear that such an operator, say A, must be anticommuting with K, i.e.

$$\{A,K\} \equiv AK + KA = 0 \qquad \textbf{(4)}$$

If at the same time this operator should commute with the Hamiltonian, then it'll generate the symmetry of the Dirac equation.

Therefore, we need an operator A with the following properties:

$$\{A,K\} = 0 \qquad \textbf{and} \qquad [A,H] = 0 \qquad \textbf{(5)}$$

It is our definition of symmetry we are looked for.

It is interesting that after this operator is constructed, we will be able to define relativistic supercharges as follows:

$$Q_1 = A, \qquad Q_2 = i\frac{AK}{|\kappa|}$$

It is obvious that

$$\{Q_1, Q_2\} = 0, \qquad Q_1^2 = Q_2^2$$

and we can find Witten's superalgebra, where $Q_i^2 \equiv h$ **is a so-called, Witten's Hamiltonian.**

Now our goal is a construction of the operator A**.We know, that there is a Dirac's** γ^5 **matrix, that anticommutes with** K**. <u>What else?</u> There is a simple <u>theorem</u>, according to which arbitrary** $(\vec{\Sigma} \cdot \vec{V})$ **type operator, where** $\vec{V}$ **is a vector with respect of** $\vec{l}$ **and is perpendicular to it,** $(\vec{l} \cdot \vec{V}) = (\vec{V} \cdot \vec{l}) = 0$**, anticommutes with** K**:**

$$\{(\vec{\Sigma} \cdot \vec{V}), K\} = 0 \qquad \textbf{(6)}$$

It is evident that the class of operators anticommuting with K **(so-called** K**-odd operators) is much wider. Any operator of the form** $\hat{O}(\vec{\Sigma} \cdot \vec{V})$**, where** $\hat{O}$ **commutes with** K**, but is otherwise arbitrary, also is a** K**-odd. This fact will be used below.**

Now one can proceed to the second stage of our problem: we wish to construct the K**-odd operator** A**, that commutes with** H**. It is clear that there remains large freedom according to the above mentioned remark about** $\hat{O}$ **operators – one can take** $\hat{O}$ **into account or ignore it.**

We have the followong physically interesting vectors at hand which obey the requirements of our theorem. They are

$\hat{\vec{r}}$ **- unit radius vector and** $\vec{p}$ **- linear momentum vector** **(7)**

Both of them are perpendicular to $\vec{l}$ *). (<u>Footnote:</u> Constraints of this theorem are also satisfied by Laplace-Runge-Lenz (LRL) vector $\vec{A} = \hat{\vec{r}} - \frac{i}{2ma}\left[\vec{p}\times\vec{l} - \vec{l}\times\vec{p}\right]$, but this vector is associated to the Coulomb potential. Hence we abstain its consideration for now).

Thus, we choose the following K-odd terms

$$\left(\vec{\Sigma}\cdot\hat{\vec{r}}\right) \quad ; \quad K\left(\vec{\Sigma}\cdot\vec{p}\right) \quad \text{and} \quad K\gamma^5 \tag{8}$$

and let probe the combination:

$$A = x_1\left(\vec{\Sigma}\cdot\hat{\vec{r}}\right) + ix_2K\left(\vec{\Sigma}\cdot\vec{p}\right) + ix_3K\gamma^5 f(r) \tag{9}$$

Here the coefficients $x_i\,(i=1,2,3)$ are choosen in such a way that A operator is hermitian for arbitrary real numbers and $f(r)$ is an arbitrary scalar function to be determined later from the symmetry requirements. Let's calculate

$$0 = [A,H] = \left(\vec{\Sigma}\cdot\hat{\vec{r}}\right)\left\{x_2V'(r) - x_3 f'(r)\right\} + \\ + 2i\beta K\gamma^5\left\{\frac{x_1}{r} - mx_3 f(r)\right\} \tag{10}$$

We have a diagonal matrix in the first row, while the antidiagonal matrix in the second one. Therefore two equations follow:

$$x_2 V'(r) = x_3 f'(r)$$
$$x_3 m f(r) = \frac{x_1}{r} \qquad \textbf{(11)}$$

One can find from these equations:

$$V(r) = \frac{x_1}{x_2}\frac{1}{mr} \qquad \textbf{(12)}$$

Therefore only the Coulomb potential corresponds to the above required $\pm\kappa$ degeneracy. The final form of obtained A operator is the following :

$$A = x_1\left\{\left(\vec{\Sigma}\cdot\hat{\vec{r}}\right) - \frac{i}{ma}K\left(\vec{\Sigma}\cdot\vec{p}\right) + \frac{i}{mr}K\gamma^5\right\}, \qquad \textbf{(13)}$$

where unessential common factor x_1 may be omited and after using known relations for Dirac matrices, this expresion may be reduced to the form

$$A = \gamma^5\left\{\vec{\alpha}\cdot\hat{\vec{r}} - \frac{i}{ma}K\gamma^5\left(H - \beta m\right)\right\} \qquad \textbf{(14)}$$

Above and here a is a strengh of Coulomb potential, $a = Z\alpha$. Precisely this operator is given in Johnson's and Lippmann's abstract.
What the real physical picture is standing behind this? Taking into account the relation

$$K\left(\vec{\Sigma}\cdot\vec{p}\right) = -i\beta\left(\vec{\Sigma}\cdot\frac{1}{2}\left[\vec{p}\times\vec{l} - \vec{l}\times\vec{p}\right]\right) \qquad \textbf{(15)}$$

one can recast our operator in the following form

$$A = \vec{\Sigma}\cdot\left(\hat{\vec{r}} - \frac{i}{2ma}\beta\left[\vec{p}\times\vec{l} - \vec{l}\times\vec{p}\right]\right) + \frac{i}{mr}K\gamma^5$$

One can see that in the non-relativistic limit, i.e. $\beta \to 1$ **and** $\gamma^5 \to 0$**, our operator reduces to**

$$A \to A_{NR} = \vec{\sigma}\cdot\left(\hat{\vec{r}} - \frac{i}{2ma}\left[\vec{p}\times\vec{l} - \vec{l}\times\vec{p}\right]\right) \qquad \textbf{(16)}$$

Note the LRL vector in the parenthesis of this equation. Therefore relativistic supercharge reduces to the projection of the LRL vector on the electron spin direction. Precisely this operator was used in the case of Pauli electron.

Because the Witten's Hamiltonian is

$$A^2 = 1 + \left(\frac{K}{a}\right)^2\left(\frac{H^2}{m^2} - 1\right) \qquad \textbf{(17)}$$

and it consists only mutually commuting operators, it is possible their simultaneous diagonalisation and replacement by corresponding eigenvalues. For instance, the energy of ground state is

$$E_0 = \left(1 - \frac{(Z\alpha)^2}{\kappa^2}\right)^{1/2} \qquad \textbf{(18)}$$

By using the ladder procedure, familiar for SUSY quantum mechanics, the Sommerfeld formula can be easily derived.

Let's remark that if we include the Lorentz-scalar potential as well

$$H = \vec{\alpha} \cdot \vec{p} + \beta m + V(r) + \beta S(r), \qquad \textbf{(19)}$$

this last Hamiltonian also commutes with K **-operator, but does not commute with the above JL operator.**

On the other hand, non-relativistic quantum mechanics is indefferent with regard of the Lorentz variance properties of potential. Therefore it is expected that in case of scalar potential the description of hidden symmetry must also be possible. In other words, the JL operator must be generalised.

For this purpose it is necessary to increase number of K **-odd structures. One has to use our theorem in the part of additional** $\hat{O}$ **factors.**

Now let us probe the following operator

$$B = x_1\left(\vec{\Sigma}\cdot\hat{\vec{r}}\right) + x_1'\left(\vec{\Sigma}\cdot\hat{\vec{r}}\right)H + ix_2K\left(\vec{\Sigma}\cdot\vec{p}\right) + ix_3K\gamma^5 f_1(r) + ix_4K\gamma^5\beta f_2(r) \qquad \textbf{(20)}$$

We included $\hat{O} = H$ **in the first structure and at the same time - the matrix** $\hat{O} = \beta$ **in the third structure. Both of them commute with** K**. It is a minimal extention of the previous picture, because only the first order structures in** $\hat{\vec{r}}$ **and** $\vec{p}$ **participate. For turning to the previous case one must take** $x_1' = x_4 = 0$ **and** $S(r) = 0$**. Calculations of relevant commutators give**

$$
\begin{aligned}
\left[B,H\right] = \gamma^5 \beta K \left\{ \frac{2ix_1}{r} - 2ix_3\left(m+S\right) f_1\left(r\right) + \frac{2ix_1'}{r} V(r) \right\} + \\
+ K\left(\vec{\Sigma}\cdot\hat{\vec{r}}\right)\left\{x_2 V'\left(r\right) - x_3 f_1'\left(r\right)\right\} + \\
+ K\beta\left(\vec{\Sigma}\cdot\hat{\vec{r}}\right)\left\{x_2 S'\left(r\right) - x_4 f_2'\left(r\right)\right\} + \\
+ \gamma^5 K \left\{ \frac{2ix_1'\left(m+S\right)}{r} - 2ix_4\left(m+S\right) f_2\left(r\right) \right\} + \\
+ \beta K \left\{ \frac{2ix_1'}{r} - 2ix_4 f_2\left(r\right) \right\} \left(\vec{\Sigma}\cdot\vec{p}\right)
\end{aligned}
\qquad \textbf{(21)}
$$

Equating this expresion to zero, we derive matrix equation, then passing to 2x2 representation we must equate to zero the coefficients standing in fronts of diagonal and antidiagonal elements. In this way it follows equations:

(1) From antidiagonal structures $\left(\gamma^5 K, \quad \gamma^5 \beta K\right)$**:**

$$
\begin{aligned}
&\frac{x_1}{r} - x_3\left(m+S\right) f_1\left(r\right) + \frac{x_1'}{r} V\left(r\right) = 0 \\
&\frac{x_1'}{r}\left(m+S\right) - \left(m+S\right) f_2\left(r\right) = 0
\end{aligned}
\qquad \textbf{(22)}
$$

(2) From diagonal structures ($K\left(\vec{\Sigma}\cdot\hat{\vec{r}}\right), \quad K\beta\left(\vec{\Sigma}\cdot\hat{\vec{r}}\right), \quad \beta K\left(\vec{\Sigma}\cdot\vec{p}\right)$):

$$
\begin{aligned}
& x_2 V'(r) - x_3 f_1'(r) = 0 \\
& \frac{x_1}{r} - x_2 (m+S)V(r) + \frac{x_1'}{r} V(r) = 0 \\
& x_2 S'(r) - x_4 f_2'(r) = 0 \\
& \frac{x_1'}{r} - x_4 f_2(r) = 0
\end{aligned}
\qquad \textbf{(23)}
$$

Integrating the first and third equations in (23) for vanishing boundary conditions at infinity, we obtain

$$
f_1(r) = \frac{x_2}{x_3} V(r), \qquad f_2(r) = \frac{x_2}{x_4} S(r) \qquad \textbf{(24)}
$$

and taking into account the last equation from (23),we have

$$f_2(r) = \frac{x_1'}{x_4 r} \tag{25}$$

Therefore, according (24) we obtain finally

$$S(r) = \frac{x_1'}{x_2 r} \tag{26}$$

So, the scalar potential must be Coulombic.
Inserting (24) into the first equation of (22) and solving for $V(r)$, one derives

$$V(r) = \frac{x_1}{r} \frac{1}{x_2(m+S) - \frac{x_1'}{r}} \tag{27}$$

At last, using here the expression (26), we find

$$V(r) = \frac{x_1}{x_2 m r} \tag{28}$$

Therefore we have asserted that the $\pm\kappa$ degeneracy is a symmetry of the Dirac equation only for Coulomb potential (for any general combination of Lorentz scalar and 4th component of a vector).

Proceed from this fact, we believe that the nature of "accidental" or "hidden" symmetry of the Kepler problem is established – it is a $\pm\kappa$ degeneracy. Generators of this symmetry, A or B describe this degeneracy – they interchange these two values. After all this by means of known ways the $SO(4)$ algebra could be constructed, manifestation of which in our macroworld is a conservation of the LRL vector and consequently closeness of celestial orbits.

Therefore, it seems that the conservation of LRL vector is a macroscopic manifestation of symmetry, which is present in microworld.

Further, if we take into account above obtained solutions, one can reduce the B operator to more compact form

$$B = \left(\vec{\Sigma}\cdot\hat{\vec{r}}\right)\left(ma_V + a_S H\right) - iK\gamma^5\left(H - \beta m\right) \qquad \textbf{(29)}$$

where the following notations are used :

$$a_V = -\frac{x_1}{x_2 m}, \qquad a_S = -\frac{x_1'}{x_2}$$

Here a_i-s are the constants of corresponding Coulomb potentials

$$V(r) = -\frac{a_V}{r}, \qquad S(r) = -\frac{a_S}{r}$$

In conclusion we want to remark, that this expression for B was derived earlier by Leviatan, who used the radial decomposition and separation of spherical angles in the Dirac equation.

Our approach is 3-dimensional, without any refering to radial equation and, therefore it is more easy and transparent. Moreover we find the source of origin of mysterious "hidden" symmetry of the Kepler problem, as the degeneracy on quantum level.