Small x processes in perturbative QCD

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Plan of the talk

- Introduction
- The gluon Reggeization
- Calculation of scattering amplitudes with all possible t–channel quantum numbers
  - Gluon channel and "bootstrap" of the gluon Reggeization
  - Colour singlet channel and the BFKL Pomeron
- BFKL and colour dipole picture
- Summary

Introduction

Cross sections of processes with a hard scale Q^2 symbolically may be written as

$$\mathcal{F} \bigotimes \hat{\sigma} \bigotimes \mathcal{F}$$

$\mathcal{F}_A^i(x, Q^2)$ —parton distributions
$\sigma_{ij}(x_i, x_j, Q^2)$ – partonic cross sections.

Evolution of the parton distributions with $\tau = \ln\left(Q^2/\Lambda^2_{QCD}\right)$ is determined by the DGLAP equations

V.N. Gribov, L.N. Lipatov, 1972,
L.N. Lipatov, 1975,
Yu.L. Dokshitzer, 1977,
G. Altarelli, Parisi, 1977

$$\frac{\partial \mathcal{F}}{\partial \tau} = \frac{\bar{\alpha}_S(Q^2)}{2} \mathcal{P} \bigotimes \mathcal{F}$$

which are basically renorm group equations. Moments of the kernels or splitting functions $P_j^i(z)$ give the anomalous dimension matrix $\gamma(N)$:

$$\gamma_{ij}(N) = \int_0^1 dz\, z^{N-1} P_j^i(z)\ .$$

The standard DGLAP approach fails at small $x = Q^2/s$ (s is c.m.s. energy squared), in particular because of the necessity to sum the terms of the perturbation series enhanced by powers of $\log(1/x)$. Resummation of leading $\log(1/x)$-terms $(\alpha_S \ln(1/x))^n$ was performed in the BFKL approach

V.S.F., E.A. Kuraev, L.N.Lipatov, 1975,
E.A. Kuraev, L.N. Lipatov, V.S.F., 1976,
Ya.Ya. Balitskii, L.N. Lipatov, 1978,

based on the gluon Reggeization.

Introduction

It describes evolution of the unintegrated gluon distribution $\mathcal{F}(x, \vec{k}^2)$ not in $\ln Q^2$, but in $\ln(1/x)$:

$$\frac{\partial \mathcal{F}}{\partial \ln(1/x)} = \mathcal{K} \bigotimes \mathcal{F},$$

$\mathcal{K}$ is the BFKL kernel and $\bigotimes$ means convolution not over fractions of longitudinal momenta as in the DGLAP equation, but over transverse momenta. The BFKL equation resums the terms
$(\alpha_S \ln(1/x))^n$ at leading order (LO$_x$),
$\alpha_S(\alpha_S \ln(1/x))^n$) at next-to-leading order (NLO$_x$).
In the leading logarithmic approximation (LLA) it predicts $\sigma \sim \left(\frac{1}{x}\right)^{\omega_P}$, where the Pomeron intercept (with subtracted 1)

$$\omega_P = 4N_c \frac{\alpha_s}{\pi} \ln 2, \quad \omega_P \simeq 0.4 \quad for \quad \alpha_s = 0.15$$

.

Introduction

The BFKL equation became famous just due to this prediction, since the rapid growth of the $\gamma^* p$ cross sections was discovered at HERA. Therefore BFKL is usually associated with the evolution equation for the unintegrated gluon distribution.

Actually the region of applicability of the BFKL approach is much wider.

The evolution equation for the unintegrated gluon distribution appears in this approach as a particular result for the imaginary part of the forward scattering amplitude ($t = 0$ and vacuum quantum numbers in the t-channel).

But the approach gives the description of scattering amplitudes at any fixed momentum transfer $\sqrt{-t}$ and at any colour state in the t-channel in the limit of large center-of-mass energy $\sqrt{s}$ (Regge limit).

Introduction

It is worthwhile to add that
the approach was developed, and is more suitable, for the description of processes with only one hard scale,
such as $\gamma^*\gamma^*$ scattering with both photon virtualities of the same order, where the DGLAP evolution is absent.
In the leading logarithmic approximation (LLA) neither scale of energy nor scale of transverse momenta entering in strong coupling $\alpha_s(k_\perp)$ are fixed. They can be determined at next-to-leading approximation NLA, when the terms

$$\alpha_S(\alpha_S \ln(1/x))^n$$

are resummed. The Pomeron intercept and normalization of cross sections can be fixed only in the NLA.

Evidently the power growth violate the Froissart bound

$$\sigma_{tot} < const(\ln s)^2.$$

This problem can not be solved by calculation of radiative corrections at any fixed $NNN...NL$ order and requires other methods. The most popular now are non-linear generalizations of the BFKL equation, related to the idea of saturation of parton densities

L.V. Gribov, E.M. Levin, M.G. Ryskin, 1983.

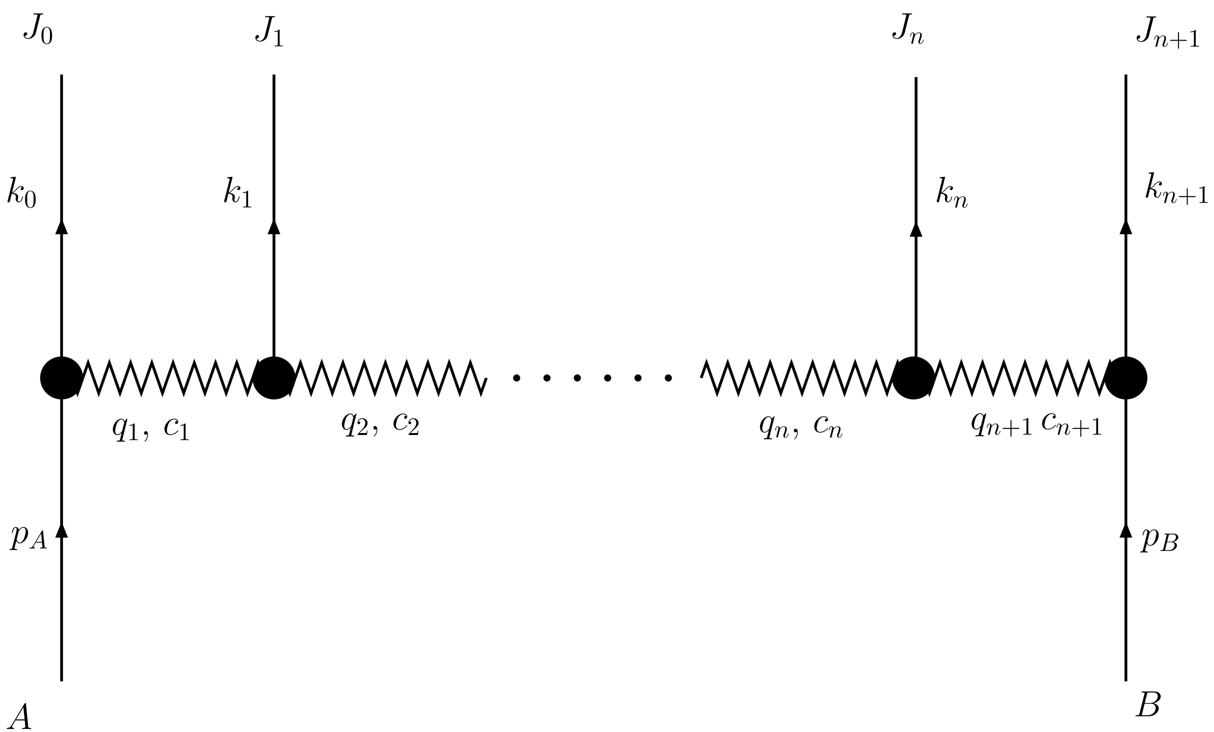
A general approach to the unitarization problem is reformulating of QCD in terms of a gauge-invariant effective field theory for the Reggeized gluon interactions

L.N. Lipatov 1995.

The gluon Reggeization

A remarkable property of QCD is the gluon Reggeization. In the multi-Regge kinematics (MRK) QCD amplitudes with the gluon exchange have the form:

$$\Re\mathcal{A}_{2\to n+2} = \bar{\Gamma}^{R_1}_{J_0 A}\left(\prod_{i=1}^{n}\frac{e^{\omega(q_i)(y_{i-1}-y_i)}}{q_{i\perp}^2}\gamma^{J_i}_{R_i R_{i+1}}\right)\frac{e^{\omega(q_{n+1})(y_n-y_{n+1})}}{q_{(n+1)\perp}^2}\Gamma^{R_{n+1}}_{J_{n+1}B}.$$



The hypothesis is extremely powerful:

- It allows us to express scattering amplitudes only through several effective vertices and gluon trajectory.
- It creates the basis of the BFKL approach to the theoretical description of high energy scattering.
- The Pomeron and Odderon in QCD appear as the compound state of the Reggeized gluons.
- The effective action based on Reggeized gluons is the most general way of the solution of saturation and unitarization problems.
- It gives a link between QCD and the String Theory.

Assuming this form the vertices $\Gamma_{P'P}$ and the Regge trajectories ω can be easily calculated in the leading order (LO).

To find them it is sufficient to calculate the simplest elastic scattering amplitude with the $P \to P'$ transition in the Born approximation. Of course, other processes can be used to test that the Regge form is valid.

To find a trajectory it is sufficient to calculate with logarithmic accuracy one-loop correction to elastic scattering amplitude with corresponding quantum numbers in the t–channel.

Of course, neither the calculation, nor the results are not so simple in the next-to-leading order (NLO).

All vertices for interaction of the Reggeon with quarks and gluons are known in the NLO

The gluon Reggeization

V.S.F., L.N. Lipatov, 1993;
V.S.F., R. Fiore, 1992;
V.S.F., R. Fiore, A. Quartarolo; 1994;
V.S.F, R. Fiore, M.I. Kotsky, 1995.

The two-loop contribution to the Regge trajectory was obtained at arbitrary space-time dimension $D = 4 + 2\epsilon$ in terms of integrals over transverse momenta

V.S.F., R. Fiore, M.I. Kotsky, 1995;
V.S.F., R. Fiore, A. Quartarolo, 1996;
V.S.F., R. Fiore, M.I. Kotsky, 1996.

The integrals can be expressed in terms of elementary functions only for $\epsilon \longrightarrow 0$.

Explicit expression for the two-loop contribution

V.S.F., M.I. Kotsky, 1996;
J. Bluemlein, V. Ravindran, W.L. van Neerven, 1998;
V.Del Duca, E.W.N. Glover, 2001

in pure gluodynamics

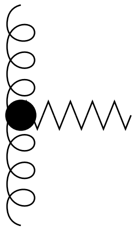
$$\omega^{(2)}(t) \simeq \left(\frac{\bar{g}^2 \left(\vec{q}^{\,2}\right)^{\epsilon}}{\epsilon} \right)^2 \left[\frac{11}{3} + \left(2\psi'(1) - \frac{67}{9} \right) \epsilon \right.$$

$$\left. + \left(\frac{404}{27} + \psi''(1) - \frac{22}{3}\psi'(1) \right) \epsilon^2 \right] , \quad \bar{g}^2 = \frac{g^2 N \Gamma(1-\epsilon)}{(4\pi)^{D/2}} .$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$, Γ is the Euler gamma-function. The space-time dimension $D = 4 + 2\epsilon \neq 4$.

The gluon Reggeization

$\Gamma^R_{Q'Q}$ and $\Gamma^R_{G'G}$ describe transitions $Q \to Q'$ and $G \to G'$ in collision with Reggeon R.



In light cone gauge the vertex of gluon transition can be written as:

$$\Gamma^{c(B)}_{G'G} = -g\big(e^*(p')e(p)\big)_\perp T^c_{G'G}$$

$$\Gamma^a_{G'G} = \Gamma^{a(B)}_{G'G}\Big\{1 + \frac{\omega^{(1)}(t)}{2}\Big[\frac{2}{\epsilon} + \psi(1) + \psi(1-\epsilon) - 2\psi(1+\epsilon) -$$

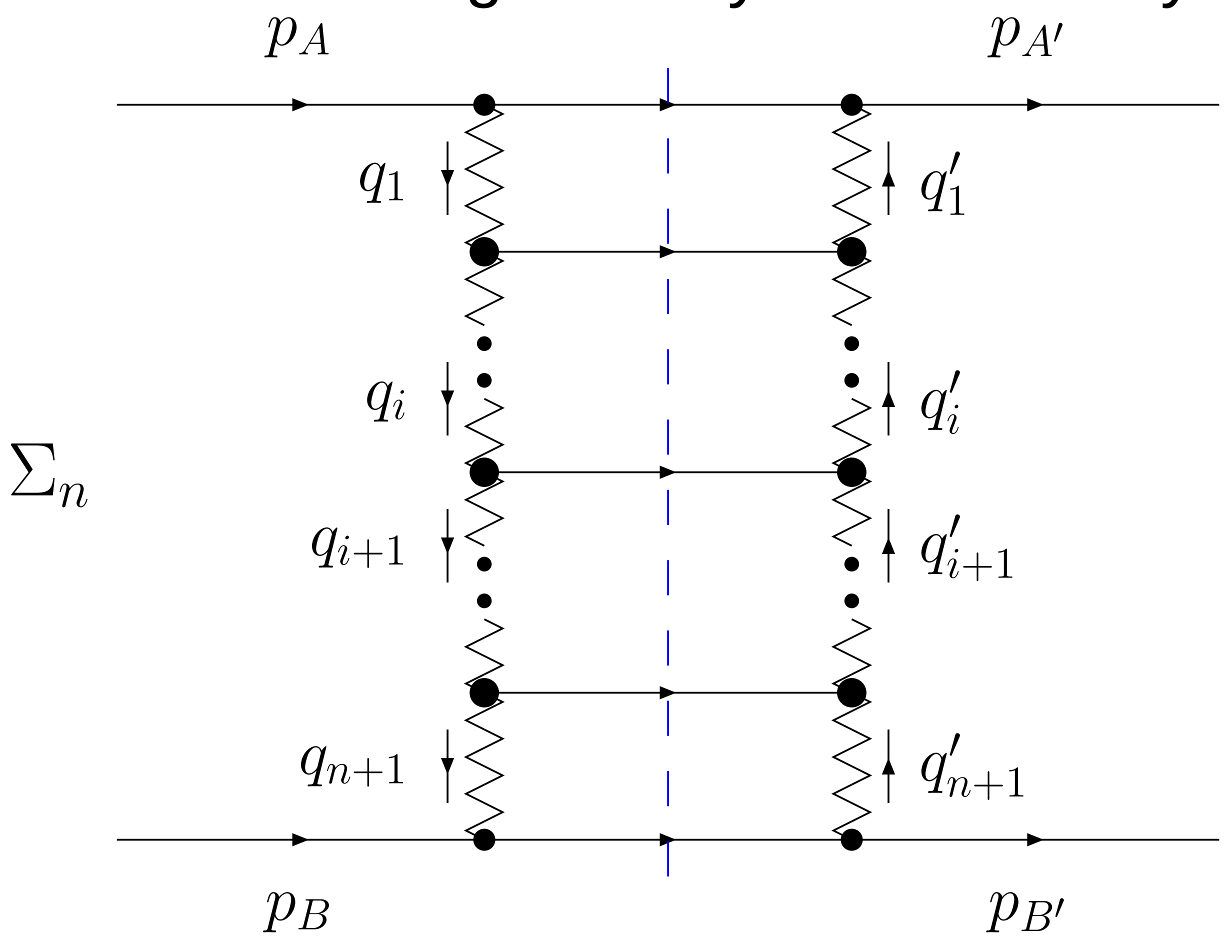
$$- \frac{9(1+\epsilon)^2+2}{2(1+\epsilon)(1+2\epsilon)(3+2\epsilon)} + \frac{n_f}{N_c}\frac{(1+\epsilon)^3+\epsilon^2}{(1+\epsilon)^2(1+2\epsilon)(3+2\epsilon)}\Big]\Big\} +$$

$$+ gT^a_{G'G}e'^*_{\perp\mu}e_{\perp\nu}\Big(g^{\mu\nu}_\perp - (D-2)\frac{q^\mu_\perp q^\nu_\perp}{q^2_\perp}\Big)\frac{\epsilon\omega^{(1)}(t)}{2(1+\epsilon)^2(1+2\epsilon)(3+2\epsilon)}\Big(1+\epsilon-\frac{n_f}{N_c}\Big),$$

V.S. F., L.N. Lipatov, 1993

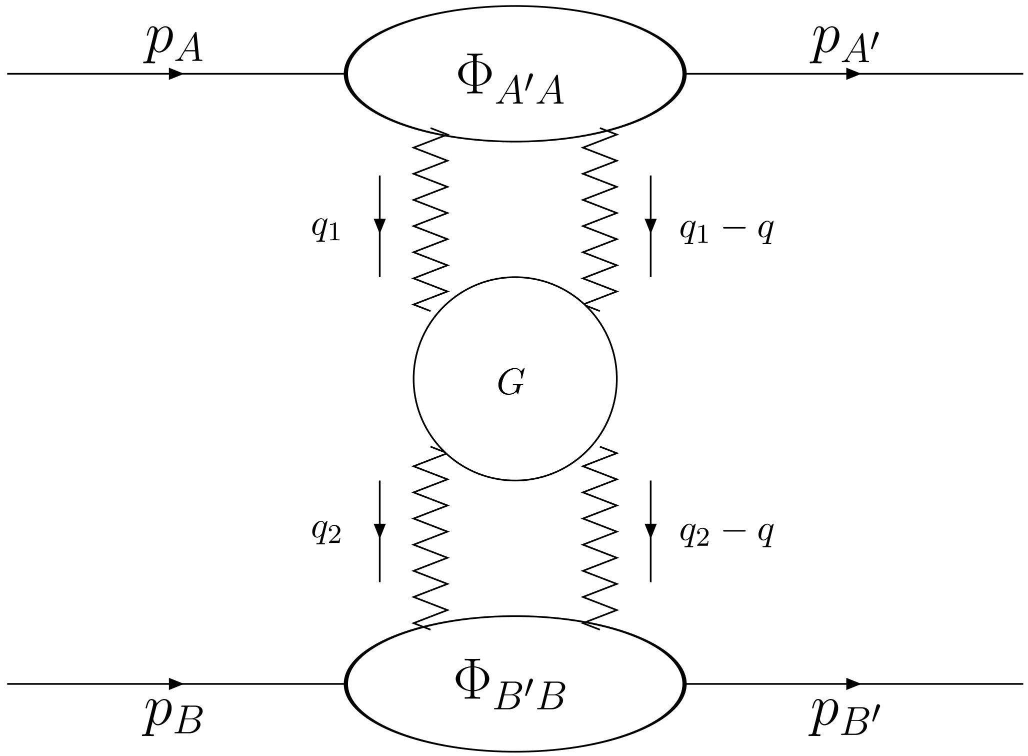
Calculation of scattering amplitudes

Amplitudes of processes with all possible quantum numbers in the t–channel are calculated using unitarity and analiticity .



The amplitudes are presented in the form :

$$\Phi_{A'A} \otimes G \otimes \Phi_{B'B}.$$



Scattering amplitudes

Impact factors $\Phi_{A'A}$ and $\Phi_{B'B}$ describe transitions $A \to A'$ $B \to B'$,
G – Green's function for two interacting Reggeized gluons,

$$\hat{\mathcal{G}} = e^{Y\hat{\mathcal{K}}},$$

$\hat{\mathcal{K}}$ – BFKL kernel, $Y = \ln(s/s_0)$,

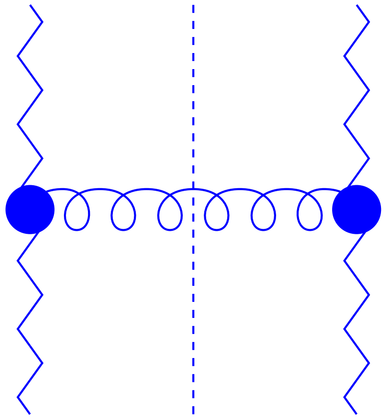
$$\hat{\mathcal{K}} = \hat{\omega}_1 + \hat{\omega}_2 + \hat{\mathcal{K}}_r$$

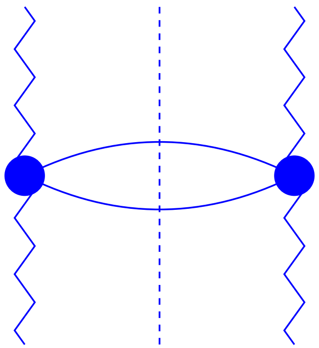
,

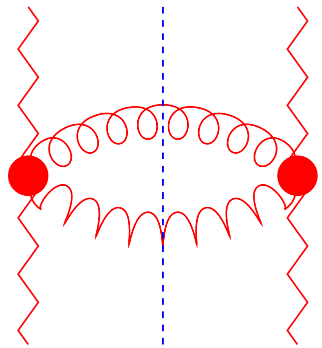
$$\hat{\mathcal{K}}_r = \hat{\mathcal{K}}_G + \hat{\mathcal{K}}_{Q\bar{Q}} + \hat{\mathcal{K}}_{GG}$$

.

Scattering amplitudes







Energy dependence of scattering amplitudes is determined by the BFKL kernel.

The gluon Reggeization

The Reggeon vertices and trajectory were obtained assuming the Reggeized form of elastic amplitudes.
This form was proved at Born level using t–channel unitarity and analyticity. Their Regeization (appearance of the Regge factors $s^{\omega(t)}$ as a result of calculation of radiative corection) arose as a hypothesis in the LLA (only gluons can be produced and each jet is actually a gluon in this approximation) on the basis of direct calculations at three-loop level for elastic amplitudes and one-loop level for one-gluon production amplitudes. Later it was proved in the LLA for all amplitudes at arbitrary number of loops with the help of bootstrap relations

Ya.Ya. Balitskii, L.N. Lipatov, V.S.F., 1978

The hypothesis is extremely powerful since an infinite number of amplitudes is expressed in terms of the gluon Regge trajectory and several Reggeon vertices.
Evidently, its proof is extremely desirable. The proof is especially necessary because of appearance of statements about existence of contributions violating the Regge ansatz at three loop level.

T. Kucs, 2004

Now the desired proof is completed

V.S.F., R. Fiore, M.G. Kozlov, A. V. Reznichenko, 2006

The gluon Reggeization

The proof of the gluon Reggeization in the NLA is also based on the bootstrap relations:

$$\frac{1}{-\pi i}\left(\sum_{l=j+1}^{n+1}\mathrm{disc}_{s_{j,l}}-\sum_{l=0}^{j-1}\mathrm{disc}_{s_{l,j}}\right)\mathcal{A}^{\mathcal{S}}_{2\to n+2}/(p_A^+p_B^-)\;=\;\frac{\partial}{\partial y_j}\,\mathcal{A}^{\mathcal{S}}_{2\to n+2}(y_i)/(p_A^+p_B^-)$$

that allow us to express partial derivatives $\partial/\partial y_j$ of the amplitudes, through the certain combination of discontinuities of the signaturized amplitudes:
$\mathcal{S}$ means symmetrization with respect to simultaneous change of signs of all $s_{i,j}$ with $i<k\le j$, performed independently for each number of channel $k=1,\ldots,n+1$.
One of the methods for the b.r. derivation is based on the Steinmann theorem in conjunction with general analytical properties of the MRK amplitudes

The gluon Reggeization

If we prove the b.r. in perturbative calculation, it will means the proof of the Regge form in NLA, since one can recursively calculate Regge amplitudes loop-by-loop in all orders of coupling constant using MRK amplitudes only in the one loop approximation for every n as an input. Indeed, b.r. express all partial derivatives of the real parts at some number of loops through the discontinuities, calculated using the s-channel unitarity in terms of amplitudes with a smaller number of loops. In the NLA only real parts of the amplitudes do contribute in the unitarity relations.

Talking about the BFKL kernel one usually has in mind the case of the forward scattering, i.e. $t = 0$ and vacuum quantum numbers in the t-channel. However, the BFKL approach is not limited to this particular case and, what is more, from the beginning it was developed for arbitrary t and for all possible t-channel colour states.
The forward BFKL kernel at NLO was found more than seven years ago.

V.S.F., L.N. Lipatov, 1998,
M. Ciafaloni, G. Camici, 1998.

The forward kernel can carry only restrictive information about the BFKL dynamics. Moreover, the non-forward case has an advantage of smaller sensitivity to large-distance contributions, since the diffusion in the infrared region is limited by $\sqrt{|t|}$. But the calculation of the non-forward kernel at NLO was completed only last year.
The reason was a complexity of the two-gluon contribution.

The “real" contribution

$$\hat{\mathcal{K}}_r = \hat{\mathcal{K}}_G + \hat{\mathcal{K}}_{Q\bar{Q}} + \hat{\mathcal{K}}_{GG}$$

is related to particle production in Reggeon-Reggeon collisions and consists of parts coming from one-gluon, two-gluon and quark-antiquark pair production. The first part is also universal, apart from a colour coefficient, and is also known in the NLO

V.S.F., D.A. Gorbachev, 2000.

The new contributions which appear in the NLO are $\hat{\mathcal{K}}_{Q\bar{Q}}$ and $\hat{\mathcal{K}}_{GG}$. Each of them is written as a sum of two terms with coefficients depending on a colour representation R in the t-channel. For the $Q\bar{Q}$ case both these terms are known. Instead, only the piece related to the gluon channel was known for the GG case.

V.S.F., D.A. Gorbachev, 2000.

Thus, the two-gluon contribution was the only missing piece in the the non-forward BFKL kernel.
The “non-subtracted" contribution to the kernel $\mathcal{K}_{GG}$ is

$$\sum_{G_1G_2}\int \gamma^{G_1G_2}\left(\gamma'^{G_1G_2}\right)^* d\phi_{G_1G_2}\ ,$$

$\gamma^{G_1G_2}$ and $\gamma'^{G_1G_2}$ – effective vertices for two-gluon production in collision of Reggeized gluons with momenta $q_1,\ -q_2$ and $q_1',\ -q_2'$ respectively;

$$q_1 - q_1' = q_2 - q_2' = q,$$

q is the total momentum transfer,

$$q_1 - q_2 = q_1' - q_2' = k_1 + k_2,$$

k_i – momenta of produced gluons,
$d\phi_{G_1G_2}$ – their phase space element; the sum is over polarizations and colours of produced gluons. For two-gluon states (and only for them) the integral over their invariant mass k^2 is logarithmically divergent at large k^2, that requires subtraction of the region of large invariant mass. This region is taken into account in the leading terms.
The two-gluon vertex

L.N. Lipatov, V.S.F., 1989.

contains two colour structures:

$$\gamma^{G_1G_2} = T^{G_1}T^{G_2}\gamma_{12} + T^{G_2}T^{G_1}\gamma_{21} \ ,$$

Accordingly, for any representation of $\mathcal{R}$ of the colour group the two-gluon contribution $\mathcal{K}^{(R)}_{GG}$ contains two terms:

"direct"

$$T^{G_1}T^{G_2}T^{G_2}T^{G_1}$$

and "interference"

$$T^{G_1}T^{G_2}T^{G_1}T^{G_2},$$

with different colour coefficients a_R and b_R and the functions F_a and F_b,

$$F_a \propto \gamma_1\gamma_1' + \gamma_2\gamma_2', \quad F_b \propto \gamma_1\gamma_2' + \gamma_2\gamma_1',$$

With account of the subtraction $\mathcal{K}_{GG}^{(R)}$ is presented in the form

$$\frac{2g^4N_c^2}{(2\pi)^{D-1}}\hat{\mathcal{S}}\int_0^1 dx \int \frac{d^{2+2\epsilon}k_1}{(2\pi)^{D-1}}\left(\frac{a_R F_a(k_1,k_2) + b_R F_b(k_1,k_2)}{x(1-x)}\right)_+ ,$$

where the operator $\hat{S}$ symmetrizes with respect to exchange of the Reggeon momenta, x is a fraction of longitudinal momenta of a produced gluon,

$$\left(\frac{f(x)}{x(1-x)}\right)_+ \equiv \frac{1}{x}[f(x)-f(0)] + \frac{1}{(1-x)}[f(x)-f(1)],$$

The group coefficients are expressed through the coefficients c_R

appearing in the leading order: $a_R = c_R^2$ and $b_R = c_R\left(c_R - \frac{1}{2}\right)$.
For the colour group $SU(N_c)$ with $N_c = 3$ the possible representations $\mathcal{R}$ are

$$\underline{1}, \underline{8_a}, \underline{8_s}, \underline{10}, \underline{\overline{10}}, \underline{27}.$$

Corresponding coefficients are

$$c_1 = 1\,,\ \ c_{8_a} = c_{8_s} = \frac{1}{2}\,,\ \ c_{10} = c_{\overline{10}} = 0\,,\ \ c_{27} = -\frac{1}{4N_c}$$

In particular,

$$a_0 = 1\ ,\ \ a_{8_a} = a_{8_s} = \frac{1}{4},\ \ b_1 = 1/2,\ \ b_{8_a} = b_{8_s} = 0.$$

The last equality is especially important for the antisymmetric case, since the vanishing of b_{8_a} is crucial for the gluon Reggeization.

The equality $b_8 = 0$ extremely simplifies calculation of the octet kernel

V.S.F., D.A. Gorbachev, 2000.

Remarkably, that only planar diagrams contribute to $\mathcal{K}_{GG}^{(8)}$ due to the colour structure.
Instead of calculation of the second term in

$$\frac{2g^4N_c^2}{(2\pi)^{D-1}}\hat{\mathcal{S}}\int_0^1 dx \int \frac{d^{2+2\epsilon}k_1}{(2\pi)^{D-1}}\left(\frac{a_R F_a(k_1,k_2)+b_R F_b(k_1,k_2)}{x(1-x)}\right)_+$$

we have found more convenient to calculate the “symmetric" contribution

$$\mathcal{K}_{GG}^{(s)}(\vec{q}_1,\vec{q}_2;\vec{q}) = \frac{2g^4N_c^2}{(2\pi)^{D-1}}\hat{\mathcal{S}}\int_0^1 dx \int \frac{d^{2+2\epsilon}k_1}{(2\pi)^{D-1}}\left(\frac{F_s(k_1,k_2)}{x(1-x)}\right)_+$$

where

$$F_s = F_a + F_b \propto (\gamma_1 + \gamma_2)(\gamma_1' + \gamma_2').$$

A marvellous feature of $\mathcal{K}_{GG}^{(s)}$ is absence of infrared singularities. The disappearance of the singularities is rather tricky: it takes place due to independence of infrared singular terms in the F_s from x. Because of this reason the singularities vanish after the substraction. Relations between the colour coefficients a_R and b_R permits to write the two-gluon contribution to the kernel for any representation R is the form

$$\mathcal{K}_{GG}^{(R)} = 2c_R\mathcal{K}_{GG}^{(8)} + b_R\mathcal{K}_{GG}^{(s)}.$$

Moreover, in pure gluodynamics an analogous relations is valid for total "real" parts of the kernel:

$$\mathcal{K}^{(R)}{}_r = 2c_R \mathcal{K}^{(8)}{}_r + b_R \mathcal{K}^{(s)}_{GG}.$$

Since $\mathcal{K}^{(s)}_{GG}$ is infrared safe, this relation greatly simplifies analysis of infrared singularities, especially because

The "real" part $\mathcal{K}^{(8)}{}_r$ for the gluon channel is rather simple

Scattering amplitudes

$$
\begin{aligned}
\mathcal{K}_r^{(8)}(\vec q_1,\vec q_2;\vec q) &= \frac{g^2 N_c}{2(2\pi)^{D-1}}\left\{\left(\frac{\vec q_1^{\,2}\vec q_2^{\,\prime\,2}+\vec q_1^{\,\prime\,2}\vec q_2^{\,2}}{\vec k^{\,2}}-\vec q^{\,2}\right)\right. \\
&\times\left(\frac{1}{2}+\frac{g^2 N_c\Gamma(1-\epsilon)(\vec k^{\,2})^{\epsilon}}{(4\pi)^{2+\epsilon}}\left(-\frac{11}{6\epsilon}+\frac{67}{18}-\zeta(2)+\epsilon\left(-\frac{202}{27}+7\zeta(3)+\frac{11}{6}\zeta(2)\right)\right)\right) \\
&+\frac{g^2 N_c\Gamma(1-\epsilon)}{(4\pi)^{2+\epsilon}}\left[\vec q^{\,2}\left(\frac{11}{6}\ln\left(\frac{\vec q_1^{\,2}\vec q_2^{\,2}}{\vec q^{\,2}\vec k^{\,2}}\right)+\frac{1}{4}\ln\left(\frac{\vec q_1^{\,2}}{\vec q^{\,2}}\right)\ln\left(\frac{\vec q_1^{\,\prime 2}}{\vec q^{\,2}}\right)+\frac{1}{4}\ln\left(\frac{\vec q_2^{\,2}}{\vec q^{\,2}}\right)\ln\left(\frac{\vec q_2^{\,\prime 2}}{\vec q^{\,2}}\right)\right.\right. \\
&\left.+\frac{1}{4}\ln^2\left(\frac{\vec q_1^{\,2}}{\vec q_2^{\,2}}\right)\right)-\frac{\vec q_1^{\,2}\vec q_2^{\,\prime\,2}+\vec q_2^{\,2}\vec q_1^{\,\prime\,2}}{2\vec k^{\,2}}\ln^2\left(\frac{\vec q_1^{\,2}}{\vec q_2^{\,2}}\right)+\frac{\vec q_1^{\,2}\vec q_2^{\,\prime\,2}-\vec q_2^{\,2}\vec q_1^{\,\prime\,2}}{\vec k^{\,2}}\ln\left(\frac{\vec q_1^{\,2}}{\vec q_2^{\,2}}\right)\left(\frac{11}{6}-\frac{1}{4}\ln\left(\frac{\vec q_1^{\,2}\vec q_2^{\,2}}{\vec k^{\,4}}\right)\right) \\
&+\frac{1}{2}[\vec q^{\,2}(\vec k^{\,2}-\vec q_1^{\,2}-\vec q_2^{\,2})+2\vec q_1^{\,2}\vec q_2^{\,2}-\vec q_1^{\,2}\vec q_2^{\,\prime\,2}-\vec q_2^{\,2}\vec q_1^{\,\prime\,2}+\frac{\vec q_1^{\,2}\vec q_2^{\,\prime\,2}-\vec q_2^{\,2}\vec q_1^{\,\prime\,2}}{\vec k^{\,2}}(\vec q_1^{\,2}-\vec q_2^{\,2})] \\
&\left.\left.\times I(\vec q_1^{\,2},\vec q_2^{\,2},\vec k^{\,2})\right]\right\}+\frac{g^2 N_c}{2(2\pi)^{D-1}}\left\{\vec q_i\longleftrightarrow\vec q_i^{\,\prime}\right\},
\end{aligned}
$$

where

$$
I(a,b,c)=\int_0^1\frac{dx}{a(1-x)+bx-cx(1-x)}\ln\left(\frac{a(1-x)+bx}{cx(1-x)}\right).
$$

The "symmetric" contribution is rather complicated. The complexity is related to the non-planar diagrams. t is known since the calculation of the non-forward kernel for the QED Pomeron

V.N. Gribov, L.N. Lipatov, G.V. Frolov, 1970

H. Cheng, T.T. Wu, 1970 where only box and cross-box diagrams are relevant. The kernel was found only in the form of two-dimensional integral.

In QCD the situation is greatly worse because of the existence of cross-pentagon and cross-hexagon diagrams in addition to QED-type cross-box diagrams.

It requires the use of additional Feynman parameters.

At arbitrary D no integration over these parameters at all can be done in elementary functions. It occurs, however, that

in the limit $\epsilon \to 0$ the integration over additional Feynman parameters can be performed, so that the result can be written as two-dimensional integral, as well as in QED.

BFKL and colour dipole picture

A very popular approach to high energy scattering is now the color dipole one

N.N. Nikolaev, B.G. Zakharov, 1991,
A.H. Mueller, 1994.

The great advantage of this approach is a clear physical interpretation in the target rest frame. Moreover, this approach is naturally applied not only at low parton density, but in the saturation regime

L.V. Gribov, E.M. Levin, M.G. Ryskin, 1983,

where equations of evolution of parton densities with energy become nonlinear. In general, there is an infinite hierarchy of coupled equations

Ia. Balitsky, 1996,
Yu. Kovchegov, 1999,
L. McLerran, R. Venugopalan, 1994,
J. Jalilian-Marian, A. Kovner, A. Leonidov, H. Weigert, 1997,
E. Iancu, A. Leonidov, L. McLerran, 2001.

BFKL and colour dipole

In the simplest case, when the target is a large nucleus, it is reduced to the BK (Balitsky-Kovchegov) equation with the kernel

$$\langle \vec{r}_1 \vec{r}_2 | \hat{\mathcal{K}}_{BK} | \vec{r}_1{}' \vec{r}_2{}' \rangle = \frac{g^2 N_c}{8\pi^3} \int d^2\rho \frac{(\vec{r}_1 - \vec{r}_2)^2}{(\vec{r}_1 - \vec{\rho})^2 (\vec{r}_2 - \vec{\rho})^2} \big(\delta(\vec{r}_1 - \vec{r}_1{}') \delta(\vec{r}_2{}'$$

$$- \vec{\rho}) + \delta(\vec{r}_2 - \vec{r}_2{}') \delta(\vec{r}_1{}' - \vec{\rho}) - \delta(\vec{r}_1 - \vec{r}_1{}') \delta(\vec{r}_2 - \vec{r}_2{}') \big)$$

It is claimed that in the linear regime the colour dipole approach gives the same results as the leading order BFKL.
But contrary to the BFKL approach, a consistent way of calculation of radiative corrections in the context of the dipole approach is not known.
Unfortunately, "native" representations for two these approaches are different.

BFKL and colour dipole

Therefore relation of these two approaches is not quite obvious. A clear understanding of this relation should help in further development of the theoretical description of small-x processes. In the leading order this relation was discussed many times. Recently it was analyzed in

J. Bartels, L. N. Lipatov, M. Salvadore, G. P. Vacca, 2005.

We

V.S.F., R. Fiore, A. Papa, 2006

have tried to extend this analysis on the NLO. The NLO generalization of the colour dipole picture should emerge as the result of this investigation. We plan to obtain both quark and gluon parts of the kernel in the dipole approach by direct transformation of the BFKL kernel in the momentum representation to the coordinate representation. Evidently, we started with the simplest part of the NLO BFKL kernel — the "non-abelian" part of the quark contribution to the kernel.

Let us begin with the leading order. Since the NLO calculations are performed using the dimensional regularization, for consistency let us use the space-time dimension $D = 4 + 2\epsilon$ in the LO as well. Then the BK kernel is

$$\langle \vec{r}_1 \vec{r}_2 | \hat{\mathcal{K}}_{BK} | \vec{r}_1{}' \vec{r}_2{}' \rangle = \frac{g^2 N_c \Gamma^2(1+\epsilon)}{8\pi^{3+2\epsilon}} \int d^{2+2\epsilon}\rho \left(\frac{(\vec{r}_1 - \vec{\rho})}{(\vec{r}_1 - \vec{\rho})^{2(1+\epsilon)}} \right.$$

$$\left. - \frac{(\vec{r}_2 - \vec{\rho})}{(\vec{r}_2 - \vec{\rho})^{2(1+\epsilon)}} \right)^2 \left(\delta(\vec{r}_1 - \vec{r}_1{}') \delta(\vec{r}_2{}' \right.$$

$$\left. - \vec{\rho}) + \delta(\vec{r}_2 - \vec{r}_2{}') \delta(\vec{r}_1{}' - \vec{\rho}) - \delta(\vec{r}_1 - \vec{r}_1{}') \delta(\vec{r}_2 - \vec{r}_2{}') \right)$$

For an irreducible representation $\mathcal{R}$ of the colour group the kernel is written as

$$\hat{\mathcal{K}}^{(\mathcal{R})} = \hat{\omega}_1 + \hat{\omega}_2 + \hat{\mathcal{K}}_r^{(\mathcal{R})},$$

In the leading order

$$\langle \vec{r}|\hat{\omega}|\vec{r}\,'\rangle = \frac{g^2 N_c \Gamma^2(1+\epsilon)}{8\pi^{3+2\epsilon}(\vec{r}-\vec{r}\,')^{2(1+2\epsilon)}}\,.$$

Therefore

$$\langle \vec{r}_1 \vec{r}_2|\hat{\omega}_1 + \hat{\omega}_2|\vec{r}_1{}' \vec{r}_2{}'\rangle$$

$$= \frac{g^2 N_c \Gamma^2(1+\epsilon)}{8\pi^{3+2\epsilon}} \left[\frac{\delta(\vec{r}_1 - \vec{r}_1{}')}{(\vec{r}_2 - \vec{r}_2{}')^{2(1+2\epsilon)}} + \frac{\delta(\vec{r}_2 - \vec{r}_2{}')}{(\vec{r}_1 - \vec{r}_1{}')^{2(1+2\epsilon)}}\right].$$

The "real" part at the leading order

BFKL and colour dipole

$$\langle\vec{r}_1\vec{r}_2|\hat{\mathcal{K}}_r^{(\mathcal{R})}|\vec{r}_1{}'\vec{r}_2{}'\rangle = \frac{g^2N_cc_{\mathcal{R}}\Gamma^2(1+\epsilon)}{4\pi^{3+2\epsilon}}\int d^{D-2}\rho\frac{(\vec{r}_1-\vec{\rho})}{(\vec{r}_1-\vec{\rho})^{2(1+\epsilon)}}\frac{(\vec{r}_2-\vec{\rho})}{(\vec{r}_2-\vec{\rho})^{2(1+\epsilon)}}$$

$$\times\left(\delta(\vec{r}_1-\vec{r}_1{}')-\delta(\vec{r}_1{}'-\vec{\rho})\right)\left(\delta(\vec{r}_2-\vec{r}_2{}')-\delta(\vec{r}_2{}'-\vec{\rho})\right)\ .$$

Therefore, for the colour singlet (Pomeron) in the t–channel we have

$$\langle\vec{r}_1\vec{r}_2|\hat{\mathcal{K}}^{(1)}|\vec{r}_1{}'\vec{r}_2{}'\rangle = \langle\vec{r}_1\vec{r}_2|\hat{\mathcal{K}}_{BK}|\vec{r}_1{}'\vec{r}_2{}'\rangle$$

$$-\frac{g^2N_c\Gamma^2(1+\epsilon)}{8\pi^{3+2\epsilon}}\left[\frac{\delta(\vec{r}_1-\vec{r}_1{}')}{(\vec{r}_1-\vec{r}_2{}')^{2(1+2\epsilon)}}+\frac{\delta(\vec{r}_2-\vec{r}_2{}')}{(\vec{r}_2-\vec{r}_1{}')^{2(1+2\epsilon)}}\right.$$

$$\left.-2\frac{\delta(\vec{r}_1{}'-\vec{r}_2{}')(\vec{r}_1-\vec{r}_1{}')(\vec{r}_2-\vec{r}_2{}')}{(\vec{r}_1-\vec{r}_1{}')^{2(1+\epsilon)}(\vec{r}_2-\vec{r}_2{}')^{2(1+\epsilon)}}\right]\ .$$

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The last term in the square brackets can be omitted supposing that the kernel acts on amplitudes possessing the "dipole property", i.e. vanishing at $\vec{r}_1{}' = \vec{r}_2{}'$; the first two terms can be omitted supposing that results of the action are convoluted with "gauge invariant", i.e. vanishing at zero momenta $\vec{q}_1$ or $\vec{q}_2$ impact factors

L. N. Lipatov, 1989,
J. Bartels, L. N. Lipatov, M. Salvadore, G. P. Vacca, 2005.

The "dipole property" permits to add to $\langle \vec{r}_1 \vec{r}_2 | \hat{\mathcal{K}}^{(1)} | \vec{r}_1{}' \vec{r}_2{}' \rangle$ terms proportional to $\delta(\vec{r}_1{}' - \vec{r}_2{}')$ (in the momentum space not depending on $\vec{q}_1{}'$ and $\vec{q}_2{}'$ separately). The "gauge invariance" permits to add to $\langle \vec{r}_1 \vec{r}_2 | \hat{\mathcal{K}}^{(1)} | \vec{r}_1{}' \vec{r}_2{}' \rangle$ terms not not depending on $\vec{r}_1$ and $\vec{r}_2$ (in the momentum space proportional to the kernel the terms proportional to $\delta(\vec{q}_1)$ or $\delta(\vec{q}_2)$).

BFKL and colour dipole

So, the BFKL and BK kernels are not related by the Fourier transform, i.e. they have only a "limited equivalence". Corresponding Green's functions are different, and only amplitudes are the same supposing the "dipole property" and "gauge invariance".
The freedom of redefinition of the kernel due to the "dipole property" of "input" amplitudes and "gauge invariance" of impact factors permits to come to the BK kernel which itself has the "dipole property", that means turns into zero at $(\vec{r}_1 = \vec{r}_2)$. Note, however, that it means

$$\langle \hat{R}_\omega | \hat{\mathcal{K}}_{BK} = 0,$$

i.e. the "bootstrap relation", which can be written as

$$\langle \hat{R}_\omega | (\hat{\mathcal{K}}^{(1)} + \hat{\omega}_1 + \hat{\omega}_2 - 2\omega(t)) = 0$$

is not valid for $\hat{\mathcal{K}}_{BK}$. It can be easily checked explicitly from

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$$\langle \vec{q}_1 \vec{q}_2 | \hat{\mathcal{K}}_{BK} | \vec{q}_1{}' \vec{q}_2{}' \rangle = \langle \vec{q}_1 \vec{q}_2 | \hat{\mathcal{K}}^{(1)} | \vec{q}_1{}' \vec{q}_2{}' \rangle$$

$$+\delta(\vec{q} - \vec{q}\,') \left[\delta(\vec{q}_2)\omega(\vec{q}_2{}') + \delta(\vec{q}_1)\omega(\vec{q}_1{}') + \frac{g^2 N_c}{(2\pi)^{3+2\epsilon}} \frac{2\vec{q}_1\vec{q}_2}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}} \right] .$$

Let us turn to the NLO. Here we consider only the quark contribution. Moreover, we use the large N_c limit, were the real contribution is strongly simplified. At arbitrary ϵ it has the form

$$\langle \vec{q}_1 \vec{q}_2 | \mathcal{K}_r^{(\hat{\mathcal{R}})Q} | \vec{q}_1{}' \vec{q}_2{}' \rangle = \delta(\vec{q} - \vec{q}\,') \frac{2 g^4 N_c n_f c_{\mathcal{R}}}{(4\pi)^{2+\epsilon}} \frac{\Gamma(1-\epsilon)}{\epsilon (2\pi)^{D-1}} \frac{\Gamma^2(2+\epsilon)}{\Gamma(4+2\epsilon)\vec{q}_1^{\,2}\vec{q}_2^{\,2}}$$

$$\times \Big\{ 2\vec{k}^{2(\epsilon-1)} (\vec{q}_1^{\,2}\vec{q}_2{}'^{\,2} + \vec{q}_2^{\,2}\vec{q}_1{}'^{\,2}) + \vec{q}^{\,2} \left(2\vec{q}^{\,2\epsilon} - \vec{q}_1^{\,2\epsilon} - \vec{q}_1{}'^{\,2\epsilon} - \vec{q}_2^{\,2\epsilon} - \vec{q}_2{}'^{\,2\epsilon} \right)$$

$$- \frac{(\vec{q}_1^{\,2}\vec{q}_2{}'^{\,2} - \vec{q}_2^{\,2}\vec{q}_1{}'^{\,2})}{\vec{k}^2} \left(\vec{q}_1^{\,2\epsilon} - \vec{q}_1{}'^{\,2\epsilon} - \vec{q}_2^{\,2\epsilon} + \vec{q}_2{}'^{\,2\epsilon} \right) \Big\} .$$

The quark contribution ω^Q to the trajectory for the case of n_f massless quark flavours can be written as

$$\langle\vec{q}|\hat{\omega}^Q|\vec{q}\,'\rangle = \delta(\vec{q}-\vec{q}\,')\frac{8g^4 N_c n_f \Gamma^2\left(1-\epsilon\right)\Gamma^2\left(2+\epsilon\right)\Gamma^2\left(1+\epsilon\right)}{(4\pi)^{4+2\epsilon}\Gamma\left(4+2\epsilon\right)\Gamma\left(1+2\epsilon\right)}(\vec{q}^{\,2})^{2\epsilon}$$

$$\times\frac{1}{\epsilon^2}\left(1-\frac{3\Gamma(1-2\epsilon)\Gamma^2(1+2\epsilon)}{2\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)\Gamma(1+3\epsilon)}\right).$$

It gives:

$$\langle\vec{r}|\hat{\omega}^Q|\vec{r}\,'\rangle = -\frac{g^4 N_c n_f \Gamma^2\left(1-\epsilon\right)\Gamma^2\left(2+\epsilon\right)\Gamma^2\left(1+\epsilon\right)\Gamma(1+3\epsilon)}{16\pi^{5+3\epsilon}\Gamma\left(4+2\epsilon\right)\Gamma(1-2\epsilon)\Gamma(1+2\epsilon)}$$

$$\times\frac{1}{\epsilon(\vec{r}-\vec{r}\,')^{2(1+3\epsilon)}}\left(1-\frac{3\Gamma(1-2\epsilon)\Gamma^2(1+2\epsilon)}{2\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)\Gamma(1+3\epsilon)}\right)$$

so that

$$\langle \vec{r}_1 \vec{r}_2 | \hat{\omega}_1^Q + \hat{\omega}_2^Q | \vec{r}_1{}' \vec{r}_2{}' \rangle = - \frac{g^4 N_c n_f \Gamma^2 \left(1-\epsilon\right) \Gamma^2 \left(2+\epsilon\right) \Gamma^2 \left(1+\epsilon\right) \Gamma(1+3\epsilon)}{16 \pi^{5+3\epsilon} \Gamma \left(4+2\epsilon\right) \Gamma(1-2\epsilon) \Gamma(1+2\epsilon)}$$

$$\times \left(1 - \frac{3\Gamma(1-2\epsilon)\Gamma^2(1+2\epsilon)}{2\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)\Gamma(1+3\epsilon)}\right) \frac{1}{\epsilon} \left[\frac{\delta(\vec{r}_1 - \vec{r}_1{}')}{(\vec{r}_2 - \vec{r}_2{}')^{2(1+3\epsilon)}} + \frac{\delta(\vec{r}_2 - \vec{r}_2{}')}{(\vec{r}_1 - \vec{r}_1{}')^{2(1+3\epsilon)}} \right].$$

For the real quark production contribution, omitting the terms with $\delta(\vec{r}_1{}' - \vec{r}_2{}')$ and using

$$\frac{1}{2\epsilon\Gamma(1+\epsilon)} \frac{1}{(\vec{r}_2 - \vec{r}_1)^{4\epsilon}} = \int \frac{d^{D-2}\rho}{\pi^{1+\epsilon}} \frac{(\vec{r}_2 - \vec{\rho})(\vec{r}_1 - \vec{\rho})}{(\vec{r}_2 - \vec{\rho})^{2(1+\epsilon)} (\vec{r}_1 - \vec{\rho})^{2(1+2\epsilon)}},$$

we obtain

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$$\langle \vec{r}_1 \vec{r}_2 | \mathcal{K}_r^{(\hat{\mathcal{R}})Q} | \vec{r}_1{}' \vec{r}_2{}' \rangle = \frac{g^4 N_c n_f c_{\mathcal{R}}}{(4\pi)^{2+\epsilon}(2\pi)^{D-1}} \frac{2^{1+4\epsilon}\Gamma^2(2+\epsilon)\Gamma(1+\epsilon))}{(3+2\epsilon)(1+\epsilon)\epsilon} \times \Bigg\{ \delta(\vec{r}_2 - \vec{r}_2{}') \int d^{D-2}\rho$$

$$\Bigg[\frac{(\vec{r}_1 - \vec{\rho})(\vec{r}_2 - \vec{\rho})}{(\vec{r}_2 - \vec{\rho})^{2(1+\epsilon)}(\vec{r}_1 - \vec{\rho})^{2(1+\epsilon)}} \left(\frac{\delta(\vec{r}_1 - \vec{r}_1{}')}{(\vec{r}_1 - \vec{\rho})^{2\epsilon}} - 2\frac{\delta(\vec{r}_1{}' - \vec{\rho})}{(\vec{r}_2 - \vec{\rho})^{2\epsilon}} - \frac{\delta(\vec{r}_1{}' - \vec{\rho})}{(\vec{r}_1 - \vec{\rho})^{2\epsilon}} - \frac{\epsilon\Gamma(1+\epsilon)}{\pi^{1+\epsilon}(\vec{r}_1{}' - \vec{\rho})^{2(1+2\epsilon)}} \right)$$

$$- \frac{1}{(\vec{r}_1 - \vec{\rho})^{2\epsilon}} \left(\frac{\delta(\vec{r}_1{}' - \vec{\rho})}{(\vec{r}_2 - \vec{\rho})^{2(1+2\epsilon)}} - \frac{\delta(\vec{r}_2 - \vec{\rho})}{(\vec{r}_1{}' - \vec{\rho})^{2(1+2\epsilon)}} \right) \Bigg] + \frac{\epsilon\Gamma(1+\epsilon)}{\pi^{1+\epsilon}} \frac{(\vec{r}_2 - \vec{r}_2{}')}{(\vec{r}_2 - \vec{r}_2{}')^{2(1+\epsilon)}}$$

$$\times \left(\frac{(\vec{r}_2{}' - \vec{r}_1{}')}{(\vec{r}_1 - \vec{r}_1{}')^{2(1+2\epsilon)}(\vec{r}_2{}' - \vec{r}_1{}')^{2(1+\epsilon)}} + \frac{(\vec{r}_1 - \vec{r}_2{}')}{(\vec{r}_1 - \vec{r}_2{}')^{2(1+\epsilon)}(\vec{r}_1 - \vec{r}_1{}')^{2(1+2\epsilon)}} \right.$$

$$\left. - \frac{(\vec{r}_1 - \vec{r}_2{}')}{(\vec{r}_1 - \vec{r}_2{}')^{2(1+\epsilon)}(\vec{r}_1{}' - \vec{r}_2{}')^{2(1+2\epsilon)}} \right) + 1 \leftrightarrow 2 \Bigg\}$$

Summary

- The BFKL approach gives the most common basis for the theoretical description of small x processes
- It is applicable to scattering amplitudes at any fixed momentum transfer $\sqrt{-t}$ and at any colour state in the t-channel
- The basis of the BFKL approach is the gluon Reggeization
- The gluon Reggeization is a remarkable property of QCD, very important for description of high energy processes
- The Reggeization hypothesis is extremely powerful: all scattering amplitudes are expressed in terms of the gluon trajectory and several Reggeon vertices

Summary

- Two steps are recently made in the development of the BFKL approach:
  - The non-forward BFKL kernel is calculated in the NLO for any colour state in the t-channel
  - The gluon Reggeization hypothesis of is proved in the NLA
- Work on search of suitable representations for the kernel and on investigation of its properties is continuing
- Particularly interesting is the NLO kernel in the coordinate representation