

Pomeron and Graviton in framework of the AdS/CFT correspondence

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1 Introduction

Pomeron contribution

$$\sigma_t \sim (s/m^2)^\Delta, \quad j_P(t) = 1 + \Delta + j'_P t$$

Pomeranchuk theorem and Odderon

$$\frac{\sigma_t^{pp}}{\sigma_t^{p\bar{p}}} \rightarrow 1, \quad \sigma_t^{pp} - \sigma_t^{p\bar{p}} \sim (s/m^2)^{\Delta_{Odd}}$$

Pomeron and Odderon as glueballs

$$P = gg, \quad O = ggg$$

Parton distributions in hadrons

$$n_r(x) = \int d^2 k_\perp n_r(x, \vec{k}_\perp)$$

DGLAP and BFKL equations

$$\frac{d n_r(x)}{d \ln Q^2} = -W_r n_r(x) + \sum_s \int_x^1 \frac{dy}{y} W_{s \rightarrow r}\left(\frac{x}{y}\right) n_s(y)$$

$$\frac{d n(x, \vec{k})}{d \ln \frac{1}{x}} = 2\omega(-\vec{k}^2) n(x, \vec{k}) + \int d^2 k' K(\vec{k}, \vec{k}') n(x, \vec{k}')$$

2 Gluon reggeization

Leading logarithmic approximation (LLA)

$$M_{AB}^{A'B'}(s, t) = s \alpha_s \sum_{n=0}^{\infty} (\alpha_s \ln s)^n a_n(t)$$

Region of applicability

$$\alpha_s \ln s \sim 1, \alpha_s = \frac{g^2}{4\pi} \rightarrow 0$$

Born amplitude

$$M_{AB}^{A'B'}(s, t)|_{Born} = g T_{A'A}^c \delta_{\lambda_{A'} \lambda_A} \frac{2s}{t} g T_{B'B}^c \delta_{\lambda_{B'} \lambda_B}$$

Gluon reggeization

$$M_{AB}^{A'B'}(s, t) = s^{\omega(t)} M_{AB}^{A'B'}(s, t)|_{Born}$$

Gluon Regge trajectory in LLA

$$\omega(-|q|^2) = -\frac{\alpha_c}{4\pi^2} N_c \int d^2k \frac{|q|^2}{|k|^2 |q-k|^2} \approx -\frac{\alpha_c}{2\pi} \ln \frac{|q|^2}{\lambda^2}$$

3 BFKL equation (1975)

Production amplitude in the MR kinematics

$$M_{2 \rightarrow 1+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2} g T_{c_2 c_1}^{d_1} C(q_2, q_1) \frac{s_2^{\omega_2}}{|q_2|^2} \dots C(q_n, q_{n-1}) \frac{s_n^{\omega_n}}{|q_n|^2}$$

Reggeon-Reggeon-gluon vertex

$$C(q_2, q_1) = \frac{q_2 q_1^*}{q_2 - q_1}$$

Impact parameter coordinates

$$\rho_k = x_k + iy_k, \quad \rho_k^* = x_k - iy_k, \quad p_k = i \frac{\partial}{\partial \rho_k}, \quad p_k^* = i \frac{\partial}{\partial \rho_k^*}$$

Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation

$$E \Psi(\vec{\rho}_1, \vec{\rho}_2) = H_{12} \Psi(\vec{\rho}_1, \vec{\rho}_2), \quad \Delta = -\frac{\alpha_s N_c}{2\pi} E$$

BFKL Hamiltonian

$$H_{12} = \ln |p_1 p_2|^2 + \frac{1}{p_1 p_2^*} \ln |\rho_{12}|^2 p_1 p_2^* \\ + \frac{1}{p_1^* p_2} \ln |\rho_{12}|^2 p_1^* p_2 - 4\psi(1), \quad \rho_{12} = \rho_1 - \rho_2$$

4 Möbius invariance (1986)

Möbius transformations

$$\rho_k \rightarrow \frac{a\rho_k + b}{c\rho_k + d_k}$$

Möbius group generators

$$M_3^{(r)} = \rho_r \partial_r, \quad M_+^{(r)} = \partial_r, \quad M_-^{(r)} = -\rho_r^2 \partial_r$$

Casimir operator for two gluons

$$M^2 = \left(\sum_{r=1}^2 \vec{M}^{(r)} \right)^2 = \rho_{12}^2 p_1 p_2, \quad \rho_{12} = \rho_1 - \rho_2$$

Eigenvalue equations

$$M^2 f_{m, \tilde{m}} = m(m-1) f_{m, \tilde{m}}, \quad M^{*2} f_{m, \tilde{m}} = \tilde{m}(\tilde{m}-1) f_{m, \tilde{m}}$$

Principal series of unitary representations

$$m = 1/2 + i\nu + n/2, \quad \tilde{m} = 1/2 + i\nu - n/2$$

5 Reggeized gluon interactions

Pomeron wave function (1986)

$$f_{m, \tilde{m}}(\vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_0) = \left(\frac{\rho_{12}}{\rho_{10} \rho_{20}} \right)^m \left(\frac{\rho_{12}^*}{\rho_{10}^* \rho_{20}^*} \right)^{\tilde{m}}$$

Pomeron energy

$$E_{m, \tilde{m}} = \varepsilon_m + \varepsilon_{\tilde{m}} \quad , \quad \varepsilon_m = \psi(m) + \psi(1 - m) - 2\psi(1)$$

BFKL Pomeron intercept

$$\Delta = 4 \frac{\alpha_s}{\pi} N_c \ln 2$$

Violation of the Froissart bound

$$\sigma \sim s^\Delta > c \ln^2 s$$

Effective Lagrangian for reggeon fields A_\pm (1995)

$$L = L_{QCD} + \text{tr} (j_- A_+ + j_+ A_-),$$

$$j_\pm = \frac{1}{g} \partial_\pm P \exp \left(g \int_{-\infty}^x v_\pm(x') dx'^\pm \right)$$

6 Holomorphic separability

Bartels-Kwiecinski-Praszalowicz equation (1980)

$$E \Psi(\vec{\rho}_1, \dots, \vec{\rho}_n) = H \Psi(\vec{\rho}_1, \dots, \vec{\rho}_n), \quad H = \sum_{k < l} \frac{\vec{T}_k \vec{T}_l}{-N_c} H_{kl}$$

Holomorphic separability at large N_c (1989)

$$H = \frac{h + h^*}{2}, \quad h = \sum_{k=1}^n h_{k,k+1}$$

Pair hamiltonian

$$h_{12} = \ln(p_1 p_2) + \frac{1}{p_1} \ln \rho_{12} p_1 + \frac{1}{p_2} \ln \rho_{12} p_2 - 2\Psi(1)$$

Holomorphic factorization (1989)

$$\Psi(\vec{\rho}_1, \vec{\rho}_2, \dots, \vec{\rho}_n) = \sum_{r,s} a_{r,s} \Psi_r(\rho_1, \dots, \rho_n) \Psi_s(\rho_1^*, \dots, \rho_n^*)$$

Singularities

$$\Psi_r(\rho_1, \dots, \rho_n) \sim \rho_{ij}^\gamma$$

7 Integrability (1993)

Two different normalization conditions

$$\|f\|_1^2 = \int \prod_{r=1}^n \frac{d^2 \rho_r}{|\rho_{r,r+1}|^2} |f|^2 ,$$

$$\|f\|_2^2 = \int \prod_{r=1}^n d^2 \rho_r \left| \prod_{r=1}^n p_r f \right|^2$$

First integral of motion

$$A = q_n = \rho_{12} \rho_{23} \dots \rho_{n1} p_1 p_2 \dots p_n , \quad [h, A] = 0$$

Transfer and monodromy matrices

$$T(u) = \text{tr } t(u) , \quad t(u) = L_1 L_2 \dots L_n = \sum_{r=0}^n u^{n-r} q_r ,$$

$$L_k = \begin{pmatrix} u + \rho_k p_k & p_k \\ -\rho_k^2 p_k & u - \rho_k p_k , \quad \hat{l} = u \hat{1} + i \hat{P} \end{pmatrix}$$

Yang-Baxter equation (1993)

$$t_{r'_1}^{s_1}(u) t_{r'_2}^{s_2}(v) l_{r'_1 r'_2}^{r'_1 r'_2}(v-u) = l_{s'_1 s'_2}^{s_1 s_2}(v-u) t_{r'_2}^{s'_2}(v) t_{r'_1}^{s'_1}(u)$$

8 Duality symmetry (1999)

Cyclic invariance at large N_c

$$\rho_i \rightarrow \rho_{i+1}, p_i \rightarrow p_{i+1}$$

Two representations of the hamiltonian

$$h = \sum_{r=1}^n (2 \ln p_r + p_r^{-1} \ln (\rho_{r,r+1} \rho_{r,r-1}) + 2\gamma) =$$

$$\sum_{r=1}^n (\rho_{r,r+1} \ln (p_r p_{r+1}) \rho_{r,r+1}^{-1} + 2 \ln \rho_{r,r+1} + 2\gamma)$$

Duality transformation

$$\rho_{r,r+1} \rightarrow p_r \rightarrow \rho_{r-1,r}$$

Duality equation

$$\psi_{m, \tilde{m}}(\overrightarrow{\rho_{12}}, \dots, \overrightarrow{\rho_{n1}}) =$$

$$|\lambda_m| \int \prod_{k=1}^{n-1} \frac{d^2 \rho'_{k-1,k}}{2\pi} \prod_{k=1}^n \frac{2e^{i\overrightarrow{\rho_{k,k+1}}} \overrightarrow{\rho'_k}}{|\rho'_{k,k+1}|^2} \psi_{\tilde{m}, m}^*(\overrightarrow{\rho'_{12}}, \dots, \overrightarrow{\rho'_{n1}})$$

9 Heisenberg spin model

Local Hamiltonian with Möbius spins \vec{M}_r

$$h = \sum_{r=1}^n h(M_{r,r+1}^2), \quad M_{r,r+1}^2 = -\rho_{r,r+1}^2 \partial_r \partial_{r+1}$$

Integrable pair interactions (L.(1994); F.,K.(1995))

$$h(M_{12}^2) = \Psi(\hat{m}_{12}) + \Psi(1 - \hat{m}_{12}) - 2\Psi(1) = h_{12}$$

Casimir operator and conformal weights

$$M_{12}^2 = -\hat{m}_{12}(1 - \hat{m}_{12})$$

Monodromy matrix parametrization

$$t(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

Pseudovacuum state and the Bethe ansatz

$$C(u)\Omega_0 = 0, \quad \Omega = \prod_k B(u_k)\Omega_0$$

10 Odderon wave function

Möbius symmetry

$$\psi(\vec{\rho}_1, \vec{\rho}_2, \vec{\rho}_3, \vec{\rho}_0) = \left(\frac{\rho_{23}}{\rho_{20}\rho_{30}} \right)^m \left(\frac{\rho_{23}^*}{\rho_{20}^*\rho_{30}^*} \right)^{\tilde{m}} \phi_{m\tilde{m}}(x, x^*)$$

Unharmonic ratio

$$x = \frac{\rho_{12} \rho_{30}}{\rho_{10} \rho_{32}}$$

Holomorphic factorization

$$\phi_{m,\tilde{m}}(x, x^*) = \sum_{r,s=1}^3 c_{rs} \phi_m^{(r)}(x) \phi_{\tilde{m}}^{(s)}(x^*)$$

Duality symmetry operator

$$a_m = x(1-x)p^{1+m}, \quad p = i \frac{d}{dx}$$

Eigenfunctions for integral of motion (1993)

$$a_m a_{1-m} \phi_m^{(r)}(x) = \lambda \phi_m^{(r)}(x)$$

11 Odderon energy

Hamiltonian in the normal form (1999)

$$\frac{h}{2} = -\log(x) - 3\psi(1) + \sum_{k=1}^{\infty} x^k f_k(P), \quad P = x\partial,$$

$$f_k(P) = -\frac{2}{k} + \frac{1}{2} \left(\frac{1}{P+k-m} + \frac{1}{P+k} \right) + \sum_{t=0}^k \frac{c_t(k)}{P+t},$$

$$c_t(k) = \frac{(-1)^{k-t} \Gamma(m+t) ((t-k)(m+t) + mk/2)}{k \Gamma(m-k+t+1) \Gamma(t+1) \Gamma(k-t+1)}$$

Wave function expansion

$$\Phi_m(P) = \sum_{n=0}^{\infty} \mu^{-n} \Phi_m^n(P), \quad \Phi_m^0(P) = 1, \quad \mu = i\lambda$$

Holomorphic energy: $\varepsilon_m/2 =$

$$\log(\mu) + 3\gamma + \left(\frac{3}{448} + \frac{13}{120} (m-1/2)^2 - \frac{(m-1/2)^4}{12} \right) \frac{1}{\mu^2}$$

$$+ \left(-\frac{4185}{2050048} - \frac{2151}{49280} (m-1/2)^2 + \dots \right) \frac{1}{\mu^4} + \dots$$

Odderon intercepts

$$\Delta^{YW} \sim -0.1 \Delta^{BFKL}; \quad \Delta^{BLV} = 0, \quad \sigma_{pp} - \sigma_{\bar{p}p} \rightarrow \text{const}$$

12 Baxter-Sklyanin picture

Pseudo-vacuum state (F.,K.)

$$\Omega_0 = \prod_{r=1}^n \rho_r^{-2}, \quad C^t(u)\Omega_0 = 0$$

Sklyanin wave function

$$\Omega = \prod_{r=1}^{n-1} Q(\hat{u}_r) \Omega_0, \quad B^t(\hat{u}) = 0$$

Transfer matrix eigenvalue equation

$$(A(u) + B(u)) \Omega = \Lambda(u) \Omega$$

Baxter equation

$$\Lambda(u) Q(u) = (u + i)^n Q(u + i) + (u - i)^n Q(u - i)$$

Eigenvalues of integrals of motion

$$\Lambda(u) = \sum_{r=0}^n \mu_r i^r u^{n-r}, \quad \mu_0 = 1, \mu_1 = 0, \mu_2 = m(m-1)$$

13 Baxter function and energy

Solutions of the Baxter equation (H.,L. ; D.,M.,K.)

$$Q^{(t)}(u) = \sum_{r=0}^{\infty} \frac{P_{t-1}(u)}{(u - ir)^t} + \sum_{r=0}^{\infty} \frac{P_{n-t-2}(u)}{(u + ir)^{n-t-1}}$$

Linear relations (V.,L.)

$$(\delta_t + \pi \coth(\pi u)) Q^{(t)}(u) = Q^{(t+1)}(u) + \alpha_t Q^{(t-1)}(u)$$

Holomorphic factorization

$$Q(\vec{u}) = \sum_{t,l=0}^{n-1} C_{t,l} Q^{(t)}(u) Q^{(l)}(u^*)$$

Energy quantization (V.,L.)

$$\epsilon = n + i \frac{d}{du} \ln(u + i)^t Q^{(t)}(u)|_{u=i}, \quad t = 1, 2, \dots, n - 1$$

Intercepts and anomalous dimensions

$$\Delta_{Odd} = 0, \quad \Delta_4; \quad \gamma_3(\omega), \quad \gamma_4(\omega)$$

14 Pomeron in a thermostat

Invariants in the t -channel

$$t = 4E^2, \quad s = -2(\vec{p})^2(1 - \cos \theta)$$

Periodicity of fields at a finite temperature

$$\phi(x_4) = \phi\left(x_4 + \frac{1}{T}\right)$$

Regge kinematics in the s -channel

$$s \gg T^2 \sim -t > 0$$

BFKL equation (V.,L.)

$$H_{12}\Psi = \Psi, \quad H_{12} = h_{12} + h_{12}^*, \quad \rho_r \rightarrow \frac{2\pi}{T} \rho_r$$

Holomorphic Hamiltonian

$$h_{12} = \sum_{r=1}^2 \left[\Omega(p_r) + \frac{1}{p_r} G(\rho_{12}) p_r \right]$$

$$\Omega(p) = \frac{\pi T}{2\lambda} + \frac{1}{2} [\psi(1 + ip) + \psi(1 - ip) - 2\psi(1)]$$

$$G(\rho_{12}) = -\frac{\pi T}{2\lambda} + \ln \left(2 \sinh \frac{\rho_{12}}{2} \right)$$

15 Integrability at finite T

Conformal transformation

$$\rho_r = \ln \rho'_r$$

Integral of motion and Hamiltonian

$$A = -(\rho'_{12})^2 \frac{\partial}{\partial \rho'_1} \frac{\partial}{\partial \rho'_2},$$

$$h_{12} = \ln(p'_1 p'_2) + \frac{1}{p'_1} \log(\rho'_{12}) p'_1 + \frac{1}{p'_2} \log(\rho'_{12}) p'_2 - 2\psi(1)$$

Operator identity

$$\frac{1}{2} \left[\psi \left(1 + z \frac{\partial}{\partial z} \right) + \psi \left(-z \frac{\partial}{\partial z} \right) \right] = \ln z + \ln \frac{\partial}{\partial z}$$

Integrable Heisenberg model with spins

$$M_k = \partial_k \quad , \quad M_+ = e^{-\rho_k} \partial_k \quad , \quad M_- = -e^{\rho_k} \partial_k$$

Baxter function for the Pomeron state (V.,L.)

$$Q(p) = \psi(p, -p)$$

16 Pomeron in $N = 4$ SUSY

BFKL kernel in the two-loop approximation (K., L.)

$$\omega = 4 \hat{a} \chi(n, \gamma) + 4 \hat{a}^2 \Delta(n, \gamma), \quad \hat{a} = g^2 N_c / (16\pi^2)$$

Non-analytic terms in QCD

$$\Delta_{QCD}(n, \gamma) = c_0 \delta_{n,0} + c_2 \delta_{n,0} + \dots$$

Hermitian separability in $N = 4$ SUSY

$$\Delta(n, \gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}, \quad M = \gamma + \frac{|n|}{2},$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2), \quad \beta'(z) = \frac{1}{4} \left[\Psi' \left(\frac{z+1}{2} \right) - \Psi' \left(\frac{z}{2} \right) \right]$$

Maximal transcendentality

$$\begin{aligned} \phi(M) = & 3\zeta(3) + \Psi''(M) - 2\Phi(M) + 2\beta'(M) \left(\Psi(1) - \Psi(M) \right) \\ & + \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M} \left(\Psi'(k+1) - \frac{\Psi(k+1) - \Psi(1)}{k+M} \right) \end{aligned}$$

17 Anomalous dimensions

Operator expansion

$$\langle \phi(\vec{\rho}_1) \phi(\vec{\rho}_2) \phi(\vec{\rho}_{1'}) \phi(\vec{\rho}_{2'}) \rangle \sim$$

$$\langle \phi(\vec{\rho}_1) \phi(\vec{\rho}_2) \sum_n |\rho_{1'2'}|^{2\Gamma_\omega(n)} \left(\frac{\rho_{1'2'}}{\rho_{1'2'}^*} \right)^{\frac{|n|}{2}} O_{|n|, \nu_\omega}(\vec{\rho}_{1'}) \rangle$$

Perturbative expansion

$$\Gamma_\omega = 1 + \frac{|n|}{2} - \gamma_n(\omega), \quad \gamma_n(\omega)|_{\omega \rightarrow 0} = \frac{g^2 N_c}{4\pi^2 \omega} + O(g^4)$$

Hypothesis (K., L. (2000))

$$\lim_{j \rightarrow -r} \gamma(j) = \lim_{|n| \rightarrow -r-1} \gamma_n(j+r)$$

Prediction (A.K., L.L. (2003))

$$\gamma(j)|_{j \rightarrow -r} = \frac{g^2 N_c}{4\pi^2} \frac{1}{j+r} + \left(\frac{g^2 N_c}{4\pi^2} \right)^2 \frac{1 + (-1)^r}{2(j+r)^3} + \dots$$

Earlier result (L.L. (1997))

$$\gamma(j) = \frac{g^2 N_c}{16\pi^2} \gamma^{LLA}(j), \quad \gamma^{LLA}(j) = 4 \left(\Psi(1) - \Psi(j-1) \right)$$

Integrable Heisenberg spin model (L.L. (1997))

One loop anomalous dimensions

Wilson twist-2 operators

$$\mathcal{O}_{\mu_1, \dots, \mu_j}^g = \hat{S} G_{\rho\mu_1}^a D_{\mu_2} D_{\mu_3} \dots D_{\mu_{j-1}} G_{\rho\mu_j}^a,$$

$$\tilde{\mathcal{O}}_{\mu_1, \dots, \mu_j}^g = \hat{S} G_{\rho\mu_1}^a D_{\mu_2} D_{\mu_3} \dots D_{\mu_{j-1}} \tilde{G}_{\rho\mu_j}^a,$$

$$\mathcal{O}_{\mu_1, \dots, \mu_j}^q = \hat{S} \bar{\Psi}^a \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_j} \Psi^a,$$

$$\tilde{\mathcal{O}}_{\mu_1, \dots, \mu_j}^q = \hat{S} \bar{\Psi}^a \gamma_5 \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_j} \Psi^a,$$

$$\mathcal{O}_{\mu_1, \dots, \mu_j}^\varphi = \hat{S} \bar{\Phi}^a D_{\mu_1} D_{\mu_2} \dots D_{\mu_j} \Phi^a$$

Anomalous dimension matrix (L.L. (1999))

$$\gamma_{gg} = -\frac{8}{j-1} + \frac{8}{j} - \frac{8}{j+1} + \frac{8}{j+2} + 8S_1(j), \quad \gamma_{q\varphi} = -\frac{16}{j},$$

$$\gamma_{gq} = -\frac{8}{j-1} + \frac{8}{j} - \frac{4}{j+1}, \quad \gamma_{g\varphi} = -\frac{8}{j-1} + \frac{8}{j},$$

$$\gamma_{qq} = -\frac{16}{j} + \frac{32}{j+1} - \frac{32}{j+2}, \quad \gamma_{q\varphi} = -\frac{16}{j} + \frac{16}{j+1} + 8S_1(j),$$

$$\gamma_{\varphi g} = -\frac{24}{j+1} + \frac{24}{j+2}, \quad \gamma_{\varphi q} = -\frac{12}{j+1}, \quad \gamma_{\varphi\varphi} = 8S_1(j),$$

$$\tilde{\gamma}_{gg} = -\frac{16}{j} + \frac{16}{j+1} + 8S_1(j), \quad \tilde{\gamma}_{gq} = -\frac{8}{j} + \frac{4}{j+1},$$

$$\tilde{\gamma}_{qq} = \frac{16}{j} - \frac{32}{j+1}, \quad \tilde{\gamma}_{q\varphi} = \frac{8}{j} - \frac{8}{j+1} + 8S_1(j)$$

Two-loop universal anomalous dimension

Diagonalization in the Born Approximation

$$\begin{vmatrix} 8S_1(j-2) & 0 & 0 \\ 0 & 8S_1(j) & 0 \\ 0 & 0 & 8S_1(j+2) \end{vmatrix},$$

$$\begin{vmatrix} 8S_1(j-1) & 0 \\ 0 & 8S_1(j+1) \end{vmatrix}$$

Universal anomalous dimension

$$U\gamma U^+ = \gamma_{uni}(j), \quad \gamma_{uni}^{(0)}(j) = -4S_1(j-2), \quad S_r(j) = \sum_{i=1}^j \frac{1}{i^r}$$

Most transcendental functions (A.K.,L.L. (2000))

$$\gamma_{uni}(j) = \hat{a}\gamma_{uni}^{(0)}(j) + \hat{a}^2\gamma_{uni}^{(1)}(j) + \hat{a}^3\gamma_{uni}^{(2)}(j) + \dots$$

Two-loop dimension (A.K.,L.L.,V.V (2003))

$$\begin{aligned} \gamma_{uni}^{(1)}(j+2)/8 &= 2S_1(j) (S_2(j)+S_{-2}(j)) - 2S_{-2,1}(j) + S_3(j) \\ &+ S_{-3}(j), \quad S_{-r}(j) = \sum_{i=1}^j \frac{(-1)^i}{i^r}, \quad S_{-2,1} = \sum_{m=1}^j \frac{(-1)^m}{m^2} S_1(m) \end{aligned}$$

Three-loop anomalous dimension

QCD dimension matrix (S.M., J.V., A. V. (2004))

$$\gamma_{QCD}^{ik}(j) = \dots, \quad i, k = g, q$$

N=4 universal dimension (A.K.,L.L.,A.O.,V.V.)

$$\begin{aligned} \gamma_{uni}^{N=4}(j) &= \hat{a} \gamma_{uni}^{(0)}(j) + \hat{a}^2 \gamma_{uni}^{(1)}(j) + \hat{a}^3 \gamma_{uni}^{(2)}(j) + \dots, \\ \gamma_{uni}^{(2)}(j+2)/32 &= -12 (S_{-3,1,1} + S_{-2,1,2} + S_{-2,2,1}) \\ &+ 6 (S_{-4,1} + S_{-3,2} + S_{-2,3}) - 3 S_{-5} - 2 S_3 S_{-2} - S_5 \\ &- 2 S_1^2 (3 S_{-3} + S_3 - 2 S_{-2,1}) - S_2 (S_{-3} + S_3 - 2 S_{-2,1}) \\ &+ 24 S_{-2,1,1,1} - S_1 (8 S_{-4} + S_{-2}^2 + 4 S_2 S_{-2} + 2 S_2^2) \\ &- S_1 (3 S_4 - 12 S_{-3,1} - 10 S_{-2,2} + 16 S_{-2,1,1}) \end{aligned}$$

Harmonic sums

$$S_a(j) = \sum_{m=1}^j \frac{1}{m^a}, \quad S_{a,b,c,\dots}(j) = \sum_{m=1}^j \frac{1}{m^a} S_{b,c,\dots}(m),$$

$$S_{-a}(j) = \sum_{m=1}^j \frac{(-1)^m}{m^a}, \quad S_{-a,b,\dots}(j) = \sum_{m=1}^j \frac{(-1)^m}{m^a} S_{b,\dots}(m),$$

$$\overline{S}_{-a,b,c,\dots}(j) = (-1)^j S_{-a,b,\dots}(j) + S_{-a,b,\dots}(\infty) \left(1 - (-1)^j\right)$$

Comparison with other approaches
(A.K.,L.L.,A.O.,V.V.)

Singularities at $j = 1 + \omega \rightarrow 0$

$$\gamma_{uni}^{N=4}(j) = \hat{a} \frac{4}{\omega} - 32\zeta_3 \hat{a}^2 + 32\zeta_3 \hat{a}^3 \frac{1}{\omega} + \dots$$

DL resummation at $j + 2r = \omega \rightarrow 0$

$$\gamma_{uni} = 4 \frac{\hat{a}}{\omega} + \frac{\gamma_{uni}^2}{\omega}$$

Anomalous dimensions at large j

$$\gamma_{uni} = a(z) \ln j, \quad z = \frac{\alpha N_c}{\pi} = 4\hat{a}$$

Perturbative results

$$a = -z + \frac{\pi^2}{12} z^2 - \frac{11}{720} \pi^4 z^3 + \dots$$

Polyakov AdS/CFT prediction

$$\lim_{z \rightarrow \infty} a = -z^{1/2} + \frac{3 \ln 2}{4\pi} + \dots$$

Resummation

$$\tilde{a} = -z + \frac{\pi^2}{12} \tilde{a}^2 = -z + \frac{\pi^2}{12} z^2 - \frac{1}{72} \pi^4 z^3 + \dots$$

Eden-Staudacher approach

Anomalous dimension at large j

$$\gamma = 8 g^2 \sigma(0) \ln j = 4 g \sqrt{2} f(0) \ln j$$

$$\sigma(t) = \epsilon f(x) , \quad t = \epsilon x , \quad \epsilon = \frac{1}{g \sqrt{2}} ,$$

Eden-Staudacher equation

$$\epsilon f(x) = \frac{t}{e^t - 1} \left(\frac{J_1(x)}{x} - \int_0^\infty dx' K(x, x') f(x') \right) ,$$

$$K(x, y) = \frac{J_1(x) J_0(y) - J_1(y) J_0(x)}{x - y} .$$

Mellin transformation

$$f(x) = \int_{-i\infty}^{i\infty} \frac{dj}{2\pi i} e^{xj} \phi(j) .$$

Ansatz for the solution

$$\phi(j) = \sum_{n=1}^{\infty} \phi_{n,\epsilon}(j) (\delta_{n,1} - a_{n,\epsilon}) ,$$

$$\phi_{n,\epsilon}(j) = \sum_{s=1}^{\infty} \frac{\left(\sqrt{(j+s\epsilon)^2 + 1} + j + s\epsilon \right)^{-n}}{\sqrt{(j+s\epsilon)^2 + 1}}$$

Analytic properties of solutions

Linear set of equations

$$a_{n,\epsilon} = \sum_{n'=1}^{\infty} K_{n,n'}(\epsilon) (\delta_{n',1} - a_{n',\epsilon}) ,$$

Kernel

$$K_{n,n'}(\epsilon) = 2n \sum_{R=0}^{\infty} (-1)^R \frac{2^{-2R-n-n'}}{\epsilon^{2R+n+n'}}$$

$$\zeta(2R+n+n') \frac{(2R+n+n'-1)! (2R+n+n')!}{R! (R+n)! (R+n')! (R+n+n')!}$$

Transcendentality with integer coefficients

$$\gamma(\epsilon) = 8 \sum_{k=1}^{\infty} \left(-\frac{1}{4\epsilon^2} \right)^k \sum_{[s_t]} c_{[s_t]} \prod_r \zeta(s_r) , \quad \sum_t s_t = 2k-2$$

Dispersion representation for solution

$$\xi(z) = \int_L \frac{dz'}{2\pi i} \frac{\xi(z') - \xi(-1/z')}{z - z'} , \quad z = j + \sqrt{j^2 + 1} ,$$

$$\frac{\xi(z) - \xi(-1/z)}{2\sqrt{j^2 + 1}} = 1 - \sum_{s=1}^{\infty} \frac{\xi\left(j + s\epsilon + \sqrt{(j + s\epsilon)^2 + 1}\right)}{\sqrt{(j + s\epsilon)^2 + 1}}$$

Pomeron and graviton at $N = 4$ SUSY (OKLV)

BFKL Pomeron in a diffusion approximation

$$j = 2 - \Delta - D\nu^2$$

Anomalous dimension of twist-2 operators

$$\gamma = 1 + \frac{j-2}{2} + i\nu$$

Constraint from the conservation of $T_{\mu\nu}$

$$\gamma = (j-2) \left(\frac{1}{2} - \frac{1/\Delta}{1 + \sqrt{1 + (j-2)/\Delta}} \right)$$

AdS/CFT for the graviton Regge trajectory

$$j = 2 + \frac{\alpha'}{2} t, \quad t = E^2/R^2, \quad \alpha' = \frac{R^2}{2} \Delta$$

Large coupling asymptotics

$$\gamma|_{z \rightarrow \infty} = -\sqrt{j-2} \Delta|_{j \rightarrow \infty}^{-1/2} = \sqrt{\pi j} z^{1/4}$$

Pomeron intercept at large α and its perturbative estimate

$$j = 2 - \Delta, \quad \Delta = \frac{1}{\pi} z^{-1/2} \approx \frac{\sqrt{3}}{2\pi} z^{-1/2}$$

Resummation of the DGLAP and BFKL equations

Slope of the anomalous dimension

$$\gamma'(2) = \frac{1}{2} - \frac{1}{2\Delta} = -\frac{\pi^2}{6}z + \frac{\pi^4}{72}z^2 - \frac{\pi^6}{540}z^3 + \dots$$

Resummation equation

$$\frac{\pi^2}{6}z = -\tilde{b} + \frac{1}{2}\tilde{b}^2, \quad b = \gamma'(2)$$

Weak and strong coupling asymptotics

$$\tilde{b} = -\frac{\pi^2}{6}z + \frac{\pi^4}{72}z^2 - \frac{\pi^6}{432}z^3 + \dots, \quad \tilde{\Delta} = \frac{\sqrt{3}}{2\pi}z^{-1/2}$$

Two-loop BFKL results

$$\omega_0 = 1 - \Delta = 4 \ln 2 z - a_1 z^2 + \dots$$

$$D = 14 \zeta(3) z - b_1 z^2 + \dots, \quad \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Resummation for large couplings

$$4 \ln 2 z = \frac{(1 - \Delta) - (2 - a_1 (4 \ln 2)^{-2}) (1 - \Delta)^2}{\Delta^2}$$

$$14 \zeta(3) z = \frac{1 - \sqrt{1 - 2 D^3 + c D^4}}{D^2}$$

18 Discussion

1. High energies and the reggeized gluon calculus.
2. Pomeron and Odderon as gluon composite states.
3. Möbius invariance of the reggeon interactions.
4. Holomorphic separability and factorization.
5. Duality symmetry and signature degeneracy.
6. Integrability of the reggeon dynamics.
7. Heisenberg model with the Möbius spins.
8. Baxter-Sklyanin representation.
9. Integrability at finite T .
10. BFKL Pomeron in $N=4$ SUSY.
11. Transcendentality for anomalous dimensions.
12. AdS/CFT correspondence and resummation.
13. Diffusion and the graviton Regge trajectory.
14. Pomeron intercept at large coupling constants.