Heat Engines: the Carnot Cycle

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Work and Heat Exchange in a Gas Expanding Reversibly

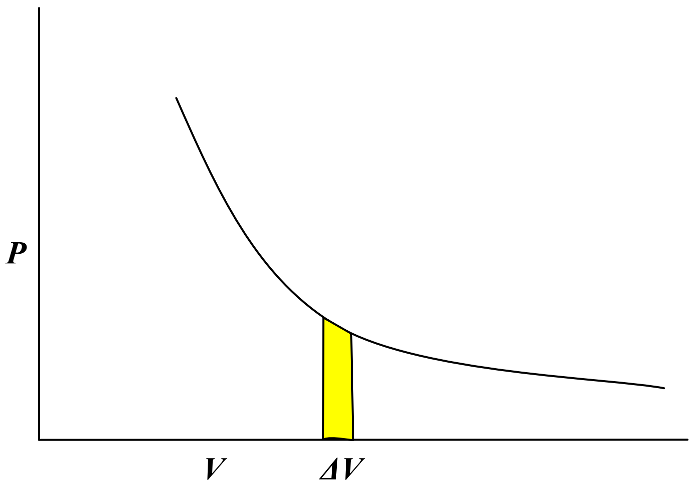
All standard heat engines (steam, gasoline, diesel) work by supplying heat to a gas, the gas then expands in a cylinder and pushes a piston to do its work. The catch is that the heat and/or the gas must somehow then be dumped out of the cylinder to get ready for the next cycle.

We examine the first step, the expansion, then go on to the full cycle—Carnot's analysis. Carnot's aim was to figure out how to maximize the efficiency of a heat engine, and then work out what that efficiency was, that is, how much of the heat supplied was actually converted into the mechanical work done by the engine. Remember that he had in mind the analogy of the water wheel, at that time still a main driving force of industry. He knew that the most efficient water wheels were those that operated smoothly, the water went into the buckets at the top from the same level, it didn't fall through any height, and didn't splash around. In the limit of a frictionless wheel, with gentle flow on and off the wheel, the machine would be reversible—turning it in reverse to raise the water back would take the same amount of work the wheel had delivered as the water fell. This was clearly perfect efficiency, so these were to conditions to emulate in the heat engine.

The analog to having the water flow into buckets at the same height, with no wasteful drop, is to have the heat from the heat supply flow into the gas at the same temperature. There must of course be a slight drop in temperature for the heat to flow at all, but this must be minimized. This means that as the heat is supplied and the gas expands, the temperature of the gas stays the same as that of the heat supply (the "heat reservoir") and the gas is expanding *isothermally*.

Isothermal Expansion

So the first question is: how much work is done by an isothermally expanding gas? Taking the temperature of the heat reservoir to be T_H (H for hot), the expanding gas follows the isothermal path $PV = nRT_H$ in the (P, V) plane.



The work done by the gas in a small volume expansion ΔV is just $P\Delta V$, the area under the curve (as we proved in the last lecture).

Hence the work done in expanding isothermally from volume V_a to V_b is the total area under the curve between those values,

$$\text{work done isothermally} = \int_{V_a}^{V_b} PdV = \int_{V_a}^{V_b} \frac{nRT_H}{V} dV = nRT_H \ln \frac{V_b}{V_a}.$$

Since the gas is at constant temperature T_H, there is no change in its internal energy during this expansion, so the total heat supplied must be $nRT_H \ln \frac{V_b}{V_a}$, the same as the external work the gas has done.

In fact, this isothermal expansion is only the first step: the gas is at the temperature of the heat reservoir, hotter than its other surroundings, and will be able to continue expanding even if the heat supply is cut off. To ensure that this further expansion is also reversible, the gas must not be losing heat to the surroundings. That is, after the heat supply is cut off, there must be no further heat exchange with the surroundings, the expansion must be *adiabatic*.

Adiabatic Expansion

The work done in an adiabatic expansion is like that done in allowing a compressed spring to expand against a force—equal to the work needed to compress the spring in the first place, for a perfect spring, and an adiabatically enclosed gas is essentially perfect in this respect. In other words, adiabatic expansion is reversible.

To find the work the gas does in expanding adiabatically from V_b to V_c, say, the above analysis is repeated with the isotherm $PV = nRT_H$ replaced by the adiabat $PV^\gamma = P_b V_b^\gamma$,

$$\text{work done adiabatically } W_{\text{adiabat}} = \int_{V_b}^{V_c} PdV = P_b V_b^{\gamma} \int_{V_b}^{V_c} \frac{dV}{V^{\gamma}} = P_b V_b^{\gamma} \frac{V_c^{1-\gamma} - V_b^{1-\gamma}}{1-\gamma}.$$

Again, this is the *area under the curve*, in this case under the adiabat, from *b* to *c* in the (*P*, *V*) plane.

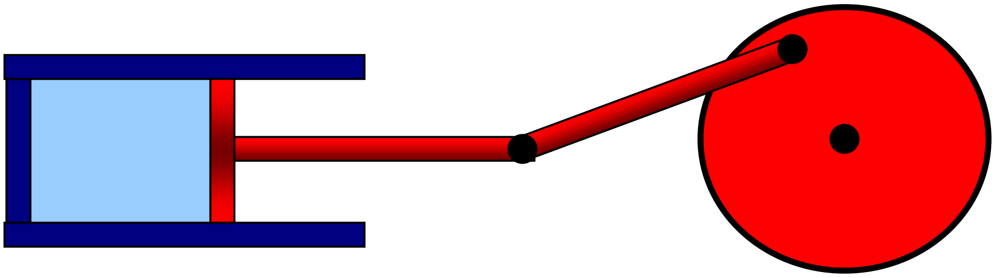
Since points *b*, *c* are on the same adiabat, $P_c V_c^{\gamma} = P_b V_b^{\gamma}$, and the expression can be written more neatly:

$$W_{\text{adiabat}} = \frac{P_c V_c - P_b V_b}{1-\gamma}.$$

This is a useful expression for the work done since we are plotting in the (*P*, *V*) plane, but note that from the gas law $PV = nRT$, the numerator is just $nR\left(T_c - T_b\right)$, and from this $W_{\text{adiabat}} = nC_V\left(T_c - T_b\right)$, as of course it must be—this is the loss of internal energy that has been expended by the gas on expanding against external pressure.

Completing the Cycle: Carnot's Ideal Heat Engine

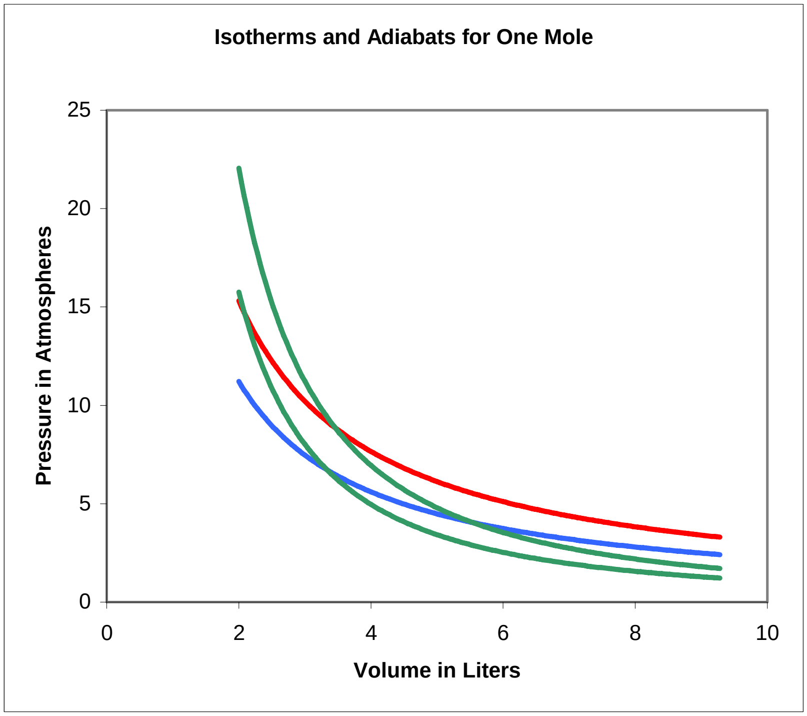
We've looked in detail at the work a gas does in expanding as heat is supplied (isothermally) and when there is no heat exchange (adiabatically). These are the two initial steps in a heat engine, but it is equally necessary for the engine to get back to where it began, for the next cycle. The general idea is that the piston drives a wheel, which continues to turn and pushes the gas back to the original volume.



But it is also essential for the gas to be as cold as possible on this return leg, because the *wheel* is now having to expend work on the *gas*, and we want that to be as little work as possible—it's costing us. The colder the gas, the less pressure the wheel is pushing against.

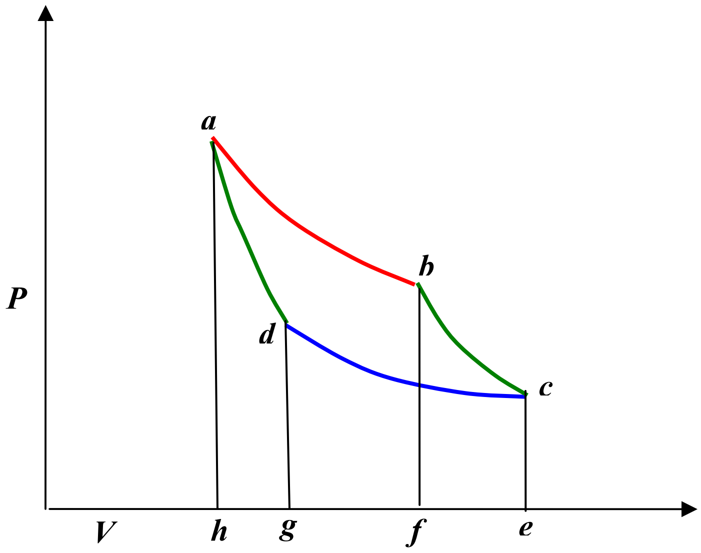
To ensure that the engine is as efficient as possible, this return path to the starting point $\left(P_a, V_a\right)$ must also be reversible. We can't just retrace the path taken in the first two legs, that would take all the work the engine did along those legs, and leave us with no net output. Now the gas cooled during the adiabatic expansion from *b* to *c*, from T_H to T_C, say, so we can go some distance back along the reversible colder isotherm T_C. But this won't get us back to $\left(P_a, V_a\right)$, because that's on the T_H isotherm. The simplest option—the one chosen by Carnot—is to proceed back along the cold isotherm to the point where it intersects the adiabat through a, then follow that isotherm back to *a*. (One could follow a more complicated path: provided it was composed of segments each being adiabatic or isothermal, it would be reversible.)

To picture the Carnot cycle in the (*P*, *V*) plane, recall from the previous lecture the graph showing two isotherms and two adiabats:



Carnot's cycle is around that curved quadrilateral having these four curves as its sides.

Let us redraw this, slightly less realistically but more conveniently:



Efficiency of the Carnot Engine

In a complete cycle of Carnot's heat engine, the gas traces the path *abcd* . The important question is: what fraction of the heat supplied from the hot reservoir (along the red top isotherm) is turned into mechanical work? This fraction is called the *efficiency* of the engine.

The work output along any curve in the (*P*, *V*) plane is just $\int PdV$ --the area under the curve, but it will be *negative* if the volume is decreasing! So the work done by the engine during the hot isothermal segment is the area *abfh*, then the adiabatic expansion adds the area *bcef*, but as the gas is compressed back, the wheel has to do work on the gas equal to the area *cdge* as heat is dumped into the cold reservoir, then *dahg* as the gas is recompressed to the starting point.

The bottom line is that the total work done by the gas is the area bounded by the four paths: the curved "parallelogram" in the picture above. We could compute this area by finding $\int PdV$ for each segment, but that is unnecessary—on completing the cycle, the gas is back to its initial temperature, so has the same internal energy. *Therefore, the work done by the engine must be just the difference between the heat supplied at* T_H *and that dumped at* T_C.

Now the heat supplies along the initial hot isothermal path *ab*, equal to the work done along that leg, is (from the paragraph above on isothermal expansion):

$$Q_H = nRT_H \ln \frac{V_b}{V_a}$$

and the heat dumped into the cold reservoir along *cd* is

$$Q_C = nRT_C \ln \frac{V_c}{V_d}.$$

The difference between these two is the net work output. This can be simplified using the adiabatic equations for the other two sides of the cycle:

$$\begin{aligned} T_H V_b^{\gamma-1} &= T_C V_c^{\gamma-1} \\ T_H V_a^{\gamma-1} &= T_C V_d^{\gamma-1}. \end{aligned}$$

Dividing the first of these equations by the second,

$$\left(\frac{V_b}{V_a}\right) = \left(\frac{V_c}{V_d}\right)$$

and using that in the preceding equation for Q_C,

$$Q_C = nRT_C \ln \frac{V_a}{V_b} = \frac{T_C}{T_H} Q_H.$$

The work done can now be written simply:

$$W = Q_H - Q_C = \left(1 - \frac{T_C}{T_H}\right) Q_H.$$

Therefore the efficiency of the engine, defined as the fraction of the ingoing heat energy that is converted to available work, is

$$\text{efficiency } = \frac{W}{Q_H} = 1 - \frac{T_C}{T_H}.$$

These temperatures are of course in degrees Kelvin, so for example the efficiency of a Carnot engine having a hot reservoir of boiling water and a cold reservoir ice cold water will be $1-(273/373)=0.27$, just over a quarter of the heat energy is transformed into useful work.

The Second Law of Thermodynamics

After all the effort to construct an efficient heat engine, making it reversible to eliminate “friction” losses, etc., it is perhaps somewhat disappointing to find this figure of 27% efficiency when operating between 0 and 100 degrees Celsius. Surely we can do better than that? After all, the heat energy of hot water is the kinetic energy of the moving molecules, can’t we find some device to channel all that energy into useful work? Well, we can do better than 27%, by having a colder cold reservoir, or a hotter hot one. But there's a limit: we can never reach 100% efficiency, because we cannot have a cold reservoir at $T_C = 0K$, and even if we did after the first cycle the heat dumped into it would warm it up!

The Second Law of Thermodynamics states that *we cannot devise an engine, working in a cycle, that simply extracts heat from a hot reservoir and delivers mechanical work.*

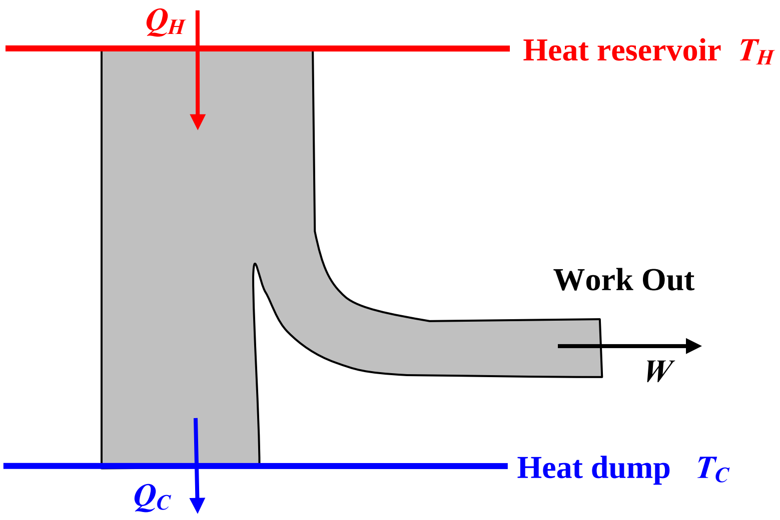
This means any engine that takes heat and delivers work also dumps out some of the initial heat to a reservoir at a lower temperature.

It's important to note that the First Law of Thermodynamics, the conservation of total energy including heat, would not be violated by an engine that powered a ship by extracting heat energy from the surrounding water. This Second Law is saying something new. And, this Second Law does not follow from the First by logical deduction—it comes (like the First) from experiment and observation.

The Most Efficient Engine

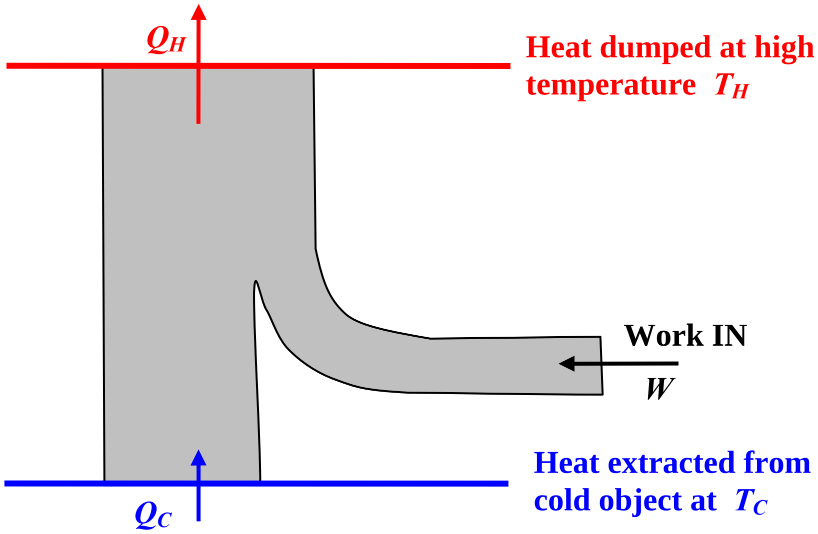
An important consequence of the second Law is that *no engine can be more efficient than the Carnot cycle*. Essentially, this is because a "super efficient" engine, if one existed, could be used to drive a Carnot cycle in reverse, which would pump back to the hot reservoir the heat the super efficient engine dumped in the cold reservoir, and the net effect of the two coupled engines would be to take heat from the hot reservoir and do work, contradicting the Second Law.

To see this, we plot the heat/energy flow for the Carnot cycle:



Here $Q_H = Q_C + W$ (all expressed in Joules, of course).

Since the engine is reversible, it can also be run backwards (this would be a refrigerator: outside work is supplied, and heat is extracted from a cold reservoir and dumped into a hot reservoir:



Suppose now we have a super efficient engine, represented by the first diagram above, and dumping the same heat per cycle Q_C into the cold reservoir, but taking in more heat energy $Q_H + \Delta$ Joules from the hot reservoir, and performing work $W + \Delta$. Now, we hook up our super efficient engine to the “Carnot refrigerator” in the diagram above. The refrigerator sucks out of the cold reservoir all the heat the super efficient engine dumped there, and needs W Joules of work per cycle to do it. The super efficient engine can provide this, and there are still Δ Joules of work to spare. Of course, the Carnot refrigerator has also dumped Q_H Joules of heat in the hot reservoir. But the bottom line is that between them, the super efficient engine and the Carnot refrigerator have extracted Δ Joules of heat from the hot reservoir and performed Δ Joules of work—contradicting the Second Law.

The Second Law therefore forces the conclusion that no amount of machine design will produce an engine more efficient than the Carnot cycle. The rather low ultimate efficiencies this dictated came as a shock to nineteenth century engineers.