From a Circling Complex Number to the Simple Harmonic Oscillator

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Describing Real Circling Motion in a Complex Way

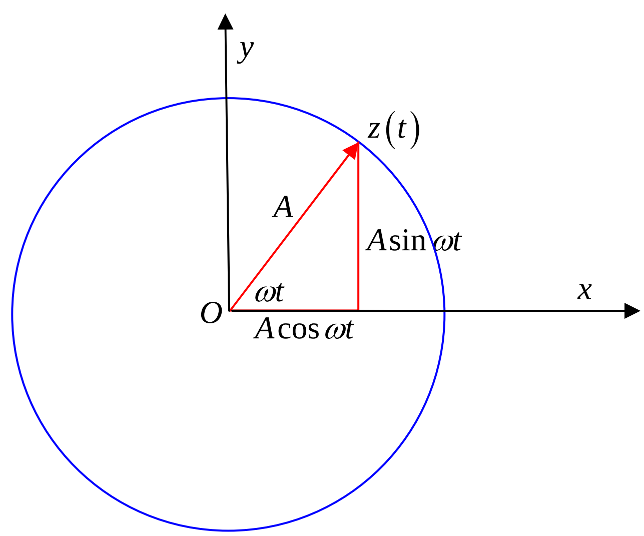
We've seen that any complex number can be written in the form $z = re^{i\theta}$, where r is the distance from the origin, and θ is the angle between a line from the origin to z and the x-axis. This means that if we have a set of numbers all with the same r, but different θ's, such as $re^{i\alpha}$, $re^{i\beta}$, etc., these are just different points on the circle with radius r centered at the origin in the complex plane.

Now think about a complex number that moves as time goes on: $z(t) = Ae^{i\omega t}$.

At time t, $z(t)$ is at a point on the circle of radius A at angle ωt to the x-axis. That is, $z(t)$ is going around the circle at a steady angular velocity ω. We can also write this:

$$z(t) = Ae^{i\omega t} = A\cos\omega t + iA\sin\omega t$$

and see that the point $z = x + iy$ is at coordinates $(x, y) = (A\cos\omega t, A\sin\omega t)$.



The angular velocity is ω, the actual velocity in the complex plane is $dz(t)/dt$.

Let's differentiate with respect to time:

$$\frac{d}{dt}Ae^{i\omega t} = \frac{d}{dt}A\left(\cos\omega t + i\sin\omega t\right) = i\omega Ae^{i\omega t} = i\omega A\left(\cos\omega t + i\sin\omega t\right) = i\omega A\cos\omega t - \omega A\sin\omega t.$$

Exercise: what are the x and y components of this velocity regarded as a vector? Show that it is perpendicular to the position vector. Why is that?

This differential equation has real and imaginary parts on both sides, so the real part on one side must be equal to the real part on the other side, and the same for imaginary parts. That gives

$$\frac{d}{dt}\cos\omega t = -\omega\sin\omega t, \quad \frac{d}{dt}\sin\omega t = \omega\cos\omega t$$

so differentiating the exponential is consistent with the standard results for trig functions.

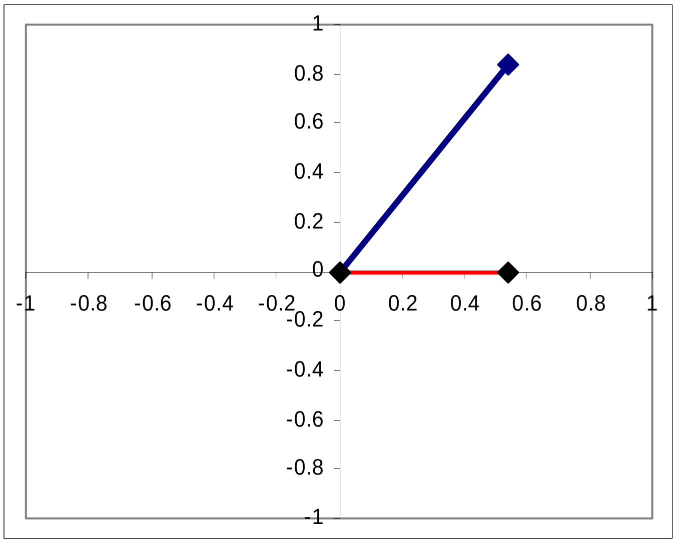
Differentiating one more time, $\frac{d^2}{dt^2}Ae^{i\omega t} = -\omega^2 Ae^{i\omega t}$.
Again going to the picture of a complex numbers as a two-dimensional vector, this is just the acceleration of an object going round in a circle of radius A at angular velocity ω, and is just $A\omega^2$ towards the center of the circle, the familiar $r\omega^2 = v^2/r$. Thinking physics here, this is the motion of an object subject to a steady central force.

Follow the Shadow: Simple Harmonic Motion

But what if we just equate the real parts of both sides? That must be a perfectly good equation: it is

$$\frac{d^2}{dt^2}A\cos\omega t = -\omega^2 A\cos\omega t$$

This is just the x-component of the circling motion, that is, *it is the "shadow" of the circling point on the x-axis*:



A simple animation of this diagram can be found here.

Forgetting for the moment about the circling point, and staring at just this *x*-axis equation, we see it describes the motion of a point having acceleration towards the origin (that is, the minus sign ensures the acceleration is in the *opposite* direction to that of the point itself from the origin) and the magnitude of the acceleration is proportional to the distance of the point from the origin.

In fact, motion of this kind is very common in nature! It is called *simple harmonic motion*.

A simple standard example is a mass hanging on a spring. If it is initially at rest, and the string has length L (stretched from its natural length to balance mg) then if it is displaced a distance x from that equilibrium position, the spring will exert an extra force $-kx$ and the equation of motion will be

$$m\frac{d^2x}{dt^2} = -kx.$$

This is *exactly* the equation of motion satisfied by the “shadow” on the *x*-axis of a point circling at a steady rate.

The general solution is $x(t) = A\cos(\omega t + \delta)$, where a possible phase δ is included so that the point can be anywhere in its oscillation at $t = 0$.

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