Lectures on Fluids

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Hydrostatics: from Archimedes to Jefferson

Beginning with Archimedes jumping out of a bath and running down the street shouting "Eureka" because he'd realized how to prove an expensive crown wasn't all it seemed, going on to his Principle of buoyancy and the concept of pressure, then to the much later realization that we live in an ocean of air with its own pressure, finally to Jefferson measuring the altitude of Monticello with a barometer bought in Philadelphia in 1776.

High Tech Crime Detection, Version 1.0

We begin this lecture in Syracuse, Sicily, 2200 years ago, with Archimedes and his friend king Heiro. The following is quoted from Vitruvius, a Roman engineer and architect writing just before the time of Christ:

> *Heiro, after gaining the royal power in Syracuse, resolved, as a consequence of his successful exploits, to place in a certain temple a golden crown which he had vowed to the immortal gods. He contracted for its making at a fixed price and weighed out a precise amount of gold to the contractor. At the appointed time the latter delivered to the king's satisfaction an exquisitely finished piece of handiwork, and it appeared that in weight the crown corresponded precisely to what the gold had weighed.*
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> *But afterwards a charge was made that gold had been abstracted and an equivalent weight of silver had been added in the manufacture of the crown. Heiro, thinking it an outrage that he had been tricked, and yet not knowing how to detect the theft, requested Archimedes to consider the matter. The latter, while the case was still on his mind, happened to go to the bath, and on getting into a tub observed that the more his body sank into it the more water ran out over the tub. As this pointed out the way to explain the case in question, without a moments delay and transported with joy, he jumped out of the tub and rushed home naked, crying in a loud voice that he had found what he was seeking; for as he ran he shouted repeatedly in Greek, "Eureka, Eureka."*
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> *Taking this as the beginning of his discovery, it is said that he made two masses of the same weight as the crown, one of gold and the other of silver. After making them, he filled a large vessel with water to the very brim and dropped the mass of silver into it. As much water ran out as was equal in bulk to that of the silver sunk in the vessel. Then, taking out the mass, he poured back the lost quantity of water, using a pint measure, until it was level with the brim as it had been before. Thus he found the weight of silver corresponding to a definite quantity of water.*

> *After this experiment, he likewise dropped the mass of gold into the full vessel and, on taking it out and measuring as before, found that not so much water was lost, but a smaller quantity: namely, as much less as a mass of gold lacks in bulk compared to a mass of silver of the same weight. Finally, filling the vessel again and dropping the crown itself into the same quantity of water, he found that more water ran over for the crown than for the mass of gold of the same weight. Hence, reasoning from the fact that more water was lost in the case of the crown than in that of the mass, he detected the mixing of silver with the gold and made the theft of the contractor perfectly clear.*

What is going on here is simply a measurement of the *density*—the mass per unit volume—of silver, gold and the crown. To measure the masses some kind of scale is used, note that at the beginning a precise amount of gold is weighed out to the contractor. Of course, if you had a nice rectangular brick of gold, and knew its weight, you wouldn't need to mess with water to determine its density, you could just figure out its volume by multiplying together length, breadth and height in meters, and divide the mass, or weight, in kilograms, by the volume to find the density in kilograms per cubic meter, or whatever units are convenient. (Actually, the original metric density measure was in grams per cubic centimeter, with the nice feature that the density of water was exactly 1, because that's how the gram was defined (at 4 degrees Celsius and atmospheric pressure, to be absolutely precise). In these units, silver has a density of 10.5, and gold of 19.3. We shall be using the standard MKS units, so water has a density 1000kg/m^3, silver 10,500kg/m^3, etc.

The problem with just trying to find the density by figuring out the volume of the crown is that it is a very complicated shape, and although one could no doubt find its volume by measuring each tiny piece and calculating a lot of small volumes which are then added together, it would take a long time and be hard to be sure of the accuracy, whereas lowering the crown into a filled bucket of water and measuring how much water overflows is obviously a pretty simple procedure. (You do have to allow for the volume of the string!). Anyway, the bottom line is that if the crown displaces more water than a block of gold of the same weight, the crown isn't pure gold.

Actually, there is one slightly surprising aspect of the story as recounted above by Vitruvius. Note that they had a weighing scale available, and a bucket suitable for immersing the crown. Given these, there was really no need to measure the amount of water slopping over. All that was necessary was to weigh the crown under water, then dry it off and weigh it out of the water. By Archimedes' Principle, the difference in these weights is equal to the weight of water displaced. This is definitely a less messy procedure--there is no need to fill the bucket to the brim in the first place, all that is necessary is to be sure that the crown is fully immersed, and not resting on the bottom or caught on the side of the bucket, during the weighing.

Of course, maybe Archimedes had not figured out his Principle when the king began to worry about the crown, perhaps the above experiment led him to it. There seems to be some confusion on this point of history.

We now turn to a discussion and derivation of Archimedes' Principle. To begin with, it is essential to understand clearly the concept of *pressure* in a fluid.

Pressure

Perhaps the simplest way to start thinking about pressure is to consider pumping up a bicycle tire with a hand pump. Pushing in the handle compresses the air inside the cylinder of the pump, raising its pressure—it gets more difficult to compress further. At a certain point, the air is compressed enough that it can get through a valve into the tire. Further pushing on the handle transfers the air to the tire, after which the handle is pulled back and the pump refills with outside air, since the valve on the tire is designed not to allow air to flow out of the tire. The outside air gets into the pump because there is, effectively, another valve—pushing the handle down pushes a washer down the inside of the cylinder to push the air out. The flexible rubber washer has a metal disk behind it so that it cannot bend far enough backwards to let air past it. On the return stroke, the washer has no rigidity, and the disk is on the wrong side for purposes of keeping it stiff, so it bends to allow air to get around it into the cylinder. As you continue to inflate the tire, it gets harder, and it's not just that you're wearing out. As the pressure in the tire increases, you have to compress the air in the pump more and more before it reaches the pressure where the valve on the tire opens and lets it in. This is hardly surprising, because the pressure inside the tire is building up, and the valve is not going to open until the pressure from the pump has built up beyond that in the tire, so that the new air can push its way into the tire.

If you look at the bicycle tire, you will probably see written on it somewhere the appropriate pressure for riding, in pounds per square inch. A gas, for instance air, under pressure pushes outwards on all the walls containing it with a steady force, equal areas of the walls feeling an equal force which is proportional to the area, hence the units, pounds per square inch. A typical ten-speed tire might be eighty pounds per square inch. (In metric units, one pound per square inch is about 6900 Newtons per square meter, so a typical ten-speed tire has pressure around 5.5×10^5 Newtons per square meter. A useful way to remember this equivalence is that the pressure of the atmosphere, which is about 15 pounds per square inch, is about 10^5 Newtons per square meter.) This means, if your pump has a cylinder with an internal cross-sectional area of one square inch, you must push it with a force of better than eighty pounds to get air into it when it's fully inflated. Of course, a pump with a narrower cylinder (which most have!) will need a proportionately smaller force, but then you will get less air delivered per stroke.

You can measure the air pressure in a tire using a pressure gauge. The basic idea of such a gauge is that on holding it against the tire valve, the valve opens and lets out air into a small cylinder with a moveable end, this end feels a force equal to the pressure in the tire (which is also the pressure in this cylinder) multiplied by its cross sectional area. The moveable end can push against a spring, with some device to lock it when it reaches maximum compression of the spring, to make it easy to read.

We can construct a more primitive, but easy to understand, pressure measuring device by putting water in a U-shaped tube. This is called a manometer. Suppose we leave one end of the tube open to the air but connect a partially inflated balloon to the other end. We observe that the pressure from the balloon pushes the water down on its side, and hence the water rises on the open side. Suppose the water goes down 0.1m on the balloon side, so it goes up 0.1m on the other side. What is the pressure in the balloon? It is clearly enough to support the weight of a column of water 0.2m high. To find out what this pressure is, we need to know how heavy water is. A cubic meter of water weighs 1000kg. Imagine this cubic meter in a cubic bucket, with a square bottom, one meter by one meter. That means the base has an area of one square meter, and this area is bearing the weight of the water, 1000kg, so the pressure at the base is approximately 10,000 Newtons per square meter. Now, our manometer, with a difference in water column heights of 0.2m, must be indicating a pressure of 2,000 Newtons per square meter. Clearly, this manometer is not going to be a handy device for measuring pressure in a ten-speed tire!

As you inflate a balloon, the elastic stretches more and more tightly, and it is this tension that balances the excess pressure of the air inside the balloon. You can feel the corresponding tightness in your own body by breathing deeply, and thus increasing the pressure in your lungs. Another way to feel increased pressure is to dive into deep water. The pressure under water increases with depth. We can easily demonstrate this with a rubber sheet stretched over a funnel on the end of a flexible rubber hose connected to the manometer. As we immerse the funnel in water, and push it in deeper, we see the pressure in it is rising, as measured by the manometer. In fact, we would expect to find that if we put it under six inches of water, the extra pressure measured will be given by an extra six inches height difference in the manometer arms. That would be true if we had a more accurate measuring instrument, but with our rather primitive arrangement the rubber sheet tension itself affects the reading slightly: as the rubber is pushed into the funnel, its own elastic contribution to the pressure inside varies. Fortunately, this turns out to be a rather small effect for the pressures we are looking at with the balloons we are using, so we ignore it.

The main point to notice is that the pressure increases *linearly* with depth, just as you might expect, because it's just the weight of water above you, weighing down on you, that's causing the pressure. But it is crucial to realize that *pressure in a fluid*, unlike the weight of a solid, *presses in all directions*. Consider again the "cubic meter" bucket filled with water we discussed previously. The pressure on the bottom was about one and a half pounds per square inch. If the water freezes to a block of ice, the weight of the ice on the bottom will exert the same pressure. However, consider now a small area on one of the side walls close to the bottom. The water presses on that too. If you're not sure about that, think what would happen if a small hole were to be bored into the side wall near the bottom. But if the water is a frozen block of ice, you could remove all the side walls, and the ice would stay in place. So fluids really are different in the way they push against containers, or against objects immersed in them.

A more vivid impression of the increase of pressure with depth, and the way it presses something in from all sides, is given by watching submarine movies, like the German das Boot, "The Boat". The submarine is forced to dive deep to avoid depth charges. As it goes down, rivets begin to pop, and water spurts in through the tiny holes. It comes in just as vigorously through a hole in the floor than through a hole in the ceiling--in fact more so, because of the greater depth. The pressure really is all around. How can fish survive under these circumstances? The main difference from this point of view between fish and submarines is that the submarine is hollow, and the pressure inside is kept down to a level tolerable to humans. The rivets pop because of the *difference* in pressure on the two sides of the sheets of steel forming the hull, which begins to buckle. In the fish, on the other hand, the fluids inside are at the same pressure as those outside. Of course, a piece of solid flesh of the fish feels this pressure all around, but it takes far greater pressure to compress a piece of solid flesh in a damaging way. (Flesh is mostly water anyway, and water is almost incompressible). For a human diver to go to great depths, the gas in the lungs must be adjusted to a pressure approximately matching the surroundings. This can be done. The problem that arises is that at high pressures, nitrogen more readily dissolves in human blood. As the diver comes back towards the surface, the pressure drops, and the nitrogen reappears in the bloodstream as small bubbles—exactly the same phenomenon as the bubbles that appear in a soda bottle when the top is loosened. The bubbles can interrupt the blood circulation, causing great pain (the “bends”) and can be fatal. So the return to surface must be gradual, to avoid a sudden appearance of large numbers of bubbles. Fish are more careful about changing depth, but apparently fish caught at considerable depth and pulled up to the surface get the bends.

Buoyancy

Consider now an object totally immersed in water, for example, a submarine. The water is pressing on its surface on all sides, top and bottom. Notice that the pressure on the bottom of the submarine (which is pressing it upwards) will be *greater* than the pressure on the top (pressing it down) just because the bottom is deeper into the water than the top is, and, as discussed above, the pressure increases linearly with the depth. Thus the total effect of the pressure forces is to tend to *lift* the submarine. This is called *buoyancy*.

To figure out how strong this buoyancy force is, imagine replacing the submarine by a *ghost* submarine—a large plastic bag filled with water, the plastic itself being extremely thin, and of negligible weight. The bag has the same size and shape as the submarine and is placed at the same depth in the water. The plastic is not stretched like a balloon, its natural size is just that of the submarine. Then this bag must feel the same pressure on each part of its surface as that on the corresponding bit of the submarine, since pressure only depends on depth, so the total buoyancy force on the bag must be the same as that on the submarine. But this bag is not going to rise or fall in the water, because it’s really just part of the water--the bag can be as thin as you like, and we could even choose a plastic having the same density as water. Since it doesn’t move, *the buoyancy force pushing it upwards must be just balanced by the weight of the water in the bag*. But we said that the submarine felt the same buoyancy force, so the upward force felt by the submari ne is *also* equal to the weight of a volume of water equal to the volume of the submarine.

This is the famous **Principle of Archimedes**:

A body immersed, or partially immersed, in a fluid is thereby acted on by an upward force of buoyancy equal to the weight of the fluid displaced.

This means that the weight of a boat, for example, must be equal to the weight of a volume of water equal to the volume of the part of the boat below the water line.

Galileo and Archimedes' Principle

Galileo fully appreciated how important Archimedes' Principle was in really understanding falling bodies of different weights, falling through media of different densities. In fact, he used it to great effect (page 66 on, Two New Sciences) to demolish Aristotle's assertion that a body ten times heavier will fall ten times faster, irrespective of the medium. Of course, this is all a little unfair to Aristotle, since Archimedes enunciated the Principle about a century after Aristotle died. The main point, which Galileo fully appreciated, is that the *weight* of a body, which is the force causing the constant downward acceleration, *must be reduced by the buoyancy force*, so the actual *total* downward force is the weight of the body minus the weight of an equal volume of the fluid. To quote Galileo (page 67, TNS):

Thus, for example, an egg made of marble will descend in water one hundred times more rapidly than a hen's egg, while in air falling from a height of twenty cubits the one will fall short of the other by less than four finger-breadths.

For this to be true, and no doubt Galileo did the experiment, the hen's egg must be about one per cent heavier than water.

Furthermore, Galileo fully realized that the same effect must be taking place in air, but the effect there is much less dramatic in general, because air (at sea level) has a density only 1/800 that of water. (Galileo thought it was 1/400). So for most objects the buoyancy force is tiny compared to the weight. The only exceptions are the bladders discussed by Galileo, in other words, balloons.

A further complication that must be borne in mind when thinking about the differing buoyancy forces in different media, and their effect on rates of fall, is that the different media also resist the motion by differing amounts. These are two quite different effects. The *buoyancy* force acts on the body even if it is at rest--it's the force that keeps ships afloat. In contrast, the *resistance*--air resistance or water resistance--is essentially *dynamic* in character, and does not depend particularly on the density of the fluid, although, of course liquids resist motion more than gases. But within those groups there are wide variations. For example, a small steel ball, say, falling through olive oil at room temperature will encounter a resistive force almost one hundred times stronger than that felt falling the same speed through water, yet olive oil is lighter than water.

To return to the buoyancy force in air, since it is about 1/800 that in water, and you are almost the same density as water, your weight as measured on a bathroom scales is lighter by about one part in 800 than your true weight. This is of course insignificant, but gives some idea how big a balloon filled with a lighter-than-air gas is needed to lift a few people. The lightest gas in nature is hydrogen, about 1/14 the density of air. Unfortunately, it is highly flammable, and early airships sometimes met a fiery end. The next lightest is helium, twice as heavy as hydrogen, but chemically completely unreactive. This is the gas used in the Goodyear blimp.

An extremely cheap alternative is just to use hot air---you can have a big bag, open at the bottom, with a heater underneath the opening. The tricky point here is the hotter the air is, the better it works, but you don’t want to set the balloon on fire! Air raised to, say, 300F, has a density about two-thirds that of air at 70F, so the buoyancy you can get with hot air is substantially below that from helium. This means the balloon has to be a lot bigger to lift the same weight. Notice also that there is a limit to how high a balloon can get. The atmosphere gets thinner with height. This means that the weight of air displaced, and hence the buoyancy force, also decreases with height, so a balloon of given density can only reach a certain height. This can be partly compensated by making a balloon of easily stretched material, so as it goes up the gas pressure inside it expands it to greater size against the lessening outside air pressure, so increasing the buoyancy.

Living in an Ocean

We discussed earlier how fish living deep in the ocean adjusted to their high pressure environment by having equal pressure, essentially, throughout their bodies, so there were no stresses, in contrast to a submarine, which has lower pressure inside than out, and so a tendency to implode. The fish are doubtless quite unaware of the fact that they live in a high pressure environment, although intelligent ones might begin to figure it out if they saw enough submarines implode. Fish control their depth by slight changes in buoyancy, achieved by moving gas in and out of an air sac, using various mechanisms: for example, by changing the acidity of the bloodstream, so that the solubility of oxygen in the blood varies.

There is actually an analogy here to our own environment. We live at the bottom of an ocean of air that covers the entire planet. Although the consequent pressure is a lot less than that at the bottom of the Atlantic, it is by no means negligible. If we pump the air out of an ordinary aluminum can it will implode. The actual pressure is about fifteen pounds per square inch. If we think of water in a U-shaped manometer tube, open to the atmosphere on both sides, the air pressure is of course equal on the water in the two arms, and the levels are the same. If we now use a pump to remove the air above the water on one side, the pressure on the water lessens, and the continuing pressure in the other arm is no longer balanced out, so the water begins to rise in the arm with less air. This is just the phenomenon of suction, the same as drinking through a straw. The important thing to see is that it is the *outside* ambient air pressure that forces the liquid up the straw.

Once it is clear that suction simply amounts to removing air pressure, and thus allowing the external air pressure of fifteen pounds per square inch, to push liquid up a pipe, it is clear that there is a limit to what suction can achieve. As the water climbs in the pipe, the pressure at the bottom of the pipe from the column of water itself increases. Eventually it will balance off the air pressure, so even if there is a perfect vacuum above the water, it won't rise any higher. It turns out that the height of a column of water that produces a pressure at its base of fifteen pounds per square inch is about thirty feet.

Galileo was actually aware of this effect, but he did not realize it was a result of limited external pressure, he thought it arose from a limit on the strength of the suction attraction holding the water together. From page 16 of TNS,

> *I once saw a cistern (a well) which had been provided with a pump under the mistaken impression that the water might thus be drawn with less effort or in greater quantity than by means of the ordinary bucket. The stock of the pump carried its sucker and valve in the upper part so that the water was lifted by attraction and not by a push as is the case with pumps in which the sucker is placed lower down. This pump worked perfectly so long as the water in the cistern stood above a certain level; but below this level the pump failed to work. When I first noticed this phenomenon I thought the machine was out of order; but the workman whom I called in to repair it told me the defect was not in the pump but in the water which had fallen too low to be raised through such a height; and he added that it was not possible, either by a pump or by any other machine working on the principle of attraction, to lift water a hair's breadth above eighteen cubits; whether the pump be large or small this is the extreme limit of the lift. Up to this time I had been so thoughtless that, although I knew a rope, or rod of wood, or of iron, if sufficiently long, would break by its own weight when held by the upper end, it never occurred to me that the same thing would happen, only much more easily, to a column of water.*

The problem with this explanation of what held solids together, as Galileo went on to admit, was that a copper wire hundreds of feet long can be hung vertically without breaking, and it is difficult to see how suction can be that much more effective for copper. We now know he was on the wrong track for once, what holds solids together is electrical forces between atoms.

Barometers

The approximately fifteen pounds per square inch pressure of the atmosphere is the pressure measured by a barometer. A column of water in a pipe with the air removed from above it would be a perfectly good barometer, just not a very handy size, since it would be about thirty feet long, from the discussion above. The obvious way to improve on this is to use a liquid heavier than water, so a less high column of it exerts the same pressure. The liquid of choice is mercury, or quicksilver, which is 13.6 times heavier than water, so we need a pipe less than three feet long. The first barometer was made by a pupil of Galileo’s, Evangelista Torricelli, and he explained how it worked in a letter written in 1644, two years after Galileo died. He wrote that the fluid rose in the pipe because of "the weight of the atmosphere", and "we live at the bottom of a sea of elemental air, which by experiment undoubtedly has weight".

The idea of using the barometer to measure heights first occurred to a Frenchman, Blaise Pascal, a few years later. He wrote an account in 1648 of taking the barometer up and down hills and tall buildings and measuring the difference in the height of the column of mercury. Of course, this is a bit tricky because the height of the column also varies with the weather, as the atmosphere sloshes about the actual amount of air above a particular point varies.

Just over a century later, on July 5, 1776, to be precise, Thomas Jefferson bought a barometer made in London at Sparhawk’s, a shop in Philadelphia, which he happened to be visiting. On September 15, 1776, he found the height of the mercury to be 29.44 inches at Monticello, 30.06 inches at the tobacco landing on the Rivanna, and 29.14 inches at the top of Montalto. He used a table published in London in 1725 to translate these differences into heights, and concluded that Montalto was 280 feet above Monticello, and 792 feet above the Rivanna.

Checking Archimedes’ Principle

Place a beaker of water about three-quarters full on a spring balance. Note the reading. Now dip your hand in the water, not touching the beaker, until the water reaches the top of he beaker. Note how much the measured weight has increased.

Question 1: If instead of putting your hand in the beaker, you had simply poured in extra water to fill it to the brim, would the measured weight have increased by the same amount, or more, or less?

Question 2: Suppose instead of dipping in your hand, you had immersed a piece of iron, (again not touching the beaker with the iron, of course) until the water reached the brim. Would this give a different reading? What if you had used a small balloon?

Take a piece of metal hanging from a spring. Note its weight, then weigh it under water in the beaker. Note also the change in weight on the spring scale under the beaker. Are these readings related? Now lower the piece of metal until it is resting on the bottom of the beaker. What do the springs read now? What happened to the buoyancy? Suppose there was a tiny spring scale on the bottom of the beaker, under the water, and the piece of metal was resting on it. What would it read?

Cartesian divers are essentially inverted test tubes, or other small containers, with trapped air inside, so that they have an overall density very close to that of water. You can put one in a container so that you can change the pressure, such as a plastic bottle with the top screwed on. If you increase the pressure, the diver dives. If you now release the pressure, it comes back up. Why?

Further comments: Carefully distinguishing between mass and density is nontrivial—after all, it took Archimedes to figure it out! Also, the units are a bit confusing. The simple everyday units would be pounds per cubic foot. The official International Units for scientific work these days are the MKS system, in which the standard length is the meter, just over a yard, and the standard weight is the kilogram, 2.2 pounds. The official unit of volume is then the cubic meter, about what a small pickup truck can carry, not a real handy size! The old official metric unit of volume was the cubic centimeter, and that was the amount of water that weighed one gram. In the new system, then, the new unit of volume is *one million* times bigger, and one cubic meter of water weighs 1,000 kilograms, which is one ton! A more handy unit of volume is the *liter*, 1,000 cubic centimeters (cc's), and a liter of water weighs one kilogram.

Question: As you know from floating in a pool, the human body has a density close to that of water. You know your own weight. What is your volume?

References

1. Vitruvius, *On Architecture IX*, Introduction 9-12, Dover Publications, NY, 1960.

2. Galileo Galilei, *Two New Sciences*, Dover Publications, NY, 1954.

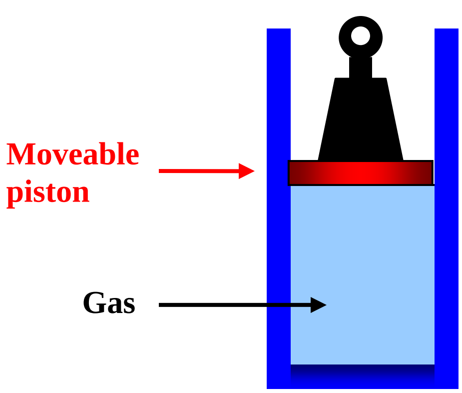
3. Silvio Bedini, *Thomas Jefferson: Statesman of Science*, Macmillan, NY, 1990.

Boyle's Law and the Law of Atmospheres

How Boyle established his famous Law PV = constant at constant temperature, and how we can use it to discover the rate of decrease of atmospheric pressure with altitude.

Introduction

We've discussed the concept of pressure in the previous lecture, introduced units of pressure (Newtons per square meter, or Pascals, and the more familiar pounds per square inch) and noted that a fluid in a container exerts pressure on all the walls, vertical as well as horizontal—if a bit of wall is removed, the fluid will squirt out.



Everyone knows that although water (like other liquids) is pretty much incompressible, air *is* compressible—you can squeeze a small balloon to a noticeably smaller volume with your hands, and you can push in a bicycle pump to some extent even if you block the end so no air escapes. . Boyle was the first person to make a *quantitative* measurement of how the volume of a fixed amount of air went down as the pressure increased.

One might imagine doing the experiment with gas in a cylinder as in the diagram here, putting on different weights and measuring the volume of the gas. Once the piston is at rest, the pressure of the gas multiplied by the area of the piston would just balance the weight of the piston plus the added weight, so the pressure is easy to find.

But there is one tricky point here: if the gas is compressed fairly rapidly—such as by adding a substantial weight, so the piston goes down suddenly—the gas heats up. Then, as the heat escapes gradually through the walls of the cylinder, the gas gradually settles into an even smaller volume.

Boyle's idea was to find out how the volume of the gas varied with outside pressure *if the temperature of the gas stayed the same*. So, if he'd done his experiment with the cylinder pictured above, he would have had to wait quite a time between volume measurements to be sure the gas was back to room temperature.

But Boyle didn't use a piston and cylinder. He did the experiment in 1662. Possibly the gun barrels manufactured at the time would have worked, with a greasy piston (I'm not sure) but he found a very elegant alternative: he trapped the air using mercury in a closed glass tube, and varied the pressure as explained below (in his own words).

He found a simple result: if the pressure was doubled, at constant temperature, the gas shrank to half its previous volume. If the pressure was tripled, it went to one-third the original volume, and so on. That is, for pressure P and volume V, at constant temperature T, PV = constant. This is *Boyle's Law*.

After reviewing Boyle's ingenious experiment, we shall see how Boyle's Law is the key to understanding a central feature of the earth's atmosphere: just how the density and pressure of air decreases with altitude. Of course, the temperature of the atmosphere also varies with height and weather, complicating the picture, but Boyle's law gives us a very good start in analyzing the situation.

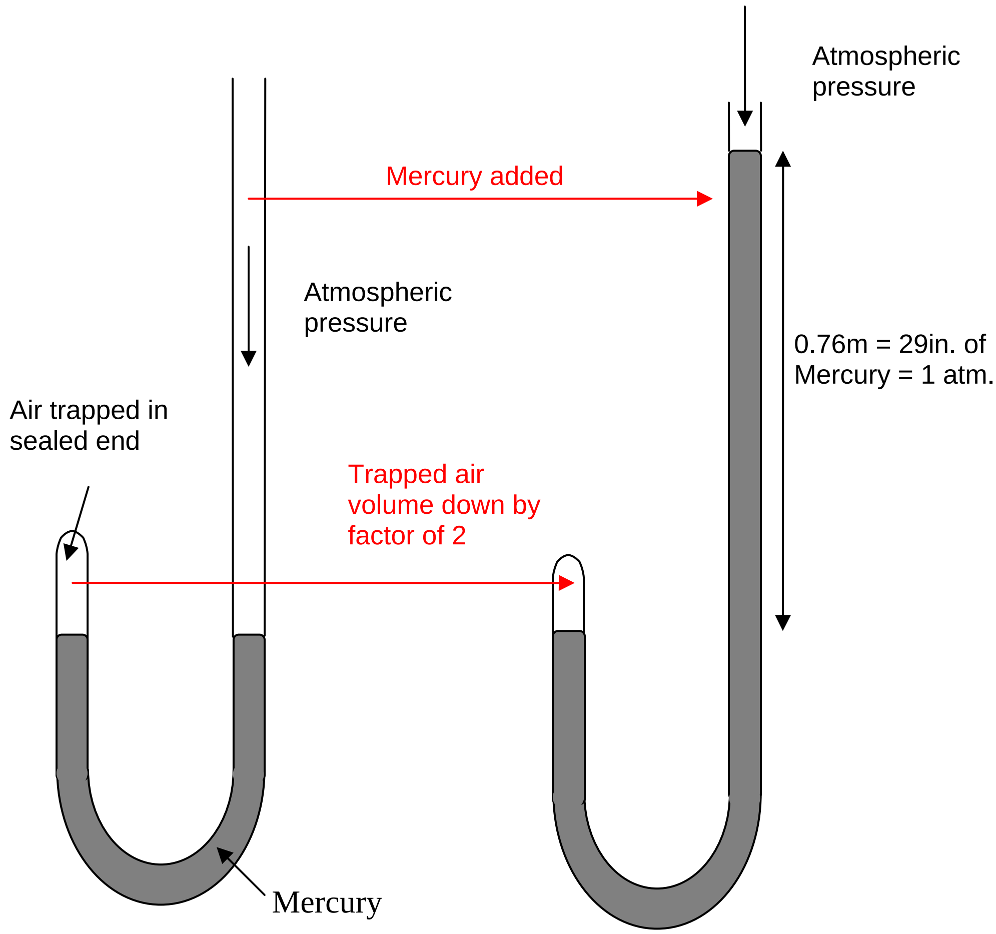
Boyle's Experiment

(*See diagram below*)

Robert Boyle was born on 1627, the fourteenth child of the Earl of Cork, an Irish landowner.
He wrote the account below in 1662. (It is from his book *A Defense of the Doctrine Touching the Spring and Weight of the Air*. I've added some notes in square brackets, which I hope clarify what's going on. Regular brackets, ( ), are Boyle's own.)

“We took then a long glass-tube, which, by a dexterous hand and the help of a lamp, [*heating it so it softens*] was in such a manner crooked at the bottom, that the part turned up was almost parallel to the rest of the tube [*they bent it into the shape in the diagram*] and the orifice of this shorter leg of the siphon (if I may so call the whole instrument) being hermetically sealed, the length of it was divided into inches (each of which was subdivided into eight parts) by a streight list of paper, which containing those divisions was carefully pasted all along it. Then putting in as much quicksilver as served to fill the arch or bended part of the siphon, that the mercury standing in a level might reach in the one leg to the bottom of the divided paper, and just to the same height of horizontal line in the other; we took care, by frequently inclining the tube, so that the air might freely pass from one leg into the other by the sides of the mercury (we took, I say, care) that the air at last included in the shorter cylinder should be of the same laxity with the rest of the air about it. [*He means at the same pressure, that is, the normal atmospheric pressure*.]

This done, we began to pour quicksilver into the longer leg of the siphon, which by its weight pressing up that in the shorter leg, did by degrees streighten [compress] the included air: and continuing this pouring in of quicksilver till the air in the shorter leg was by condensation reduced to take up by half the space it possessed before; we cast our eyes upon the longer leg of the glass, on which was likewise pasted a list of paper carefully divided into inches and parts, and we observed, not without delight and satisfaction, that the quicksilver in that longer part of the tube was twenty-nine inches higher than the other.”



Boyle found that when more mercury was poured into the tube, increasing pressure on the trapped air, the air volume halved if the *total* pressure, including that from the atmosphere, was doubled.

Boyle's "delight and satisfaction" in that last sentence arose because he knew that the extra pressure exerted by the added twenty-nine inches of mercury was equal to an extra atmosphere, so the air trapped in the shorter tube had *halved* in volume when the pressure was *doubled*. He went on the repeat the experiment many times, with different heights of the column of mercury in the longer tube, and checking each day on the actual atmospheric pressure at the time of the experiment.

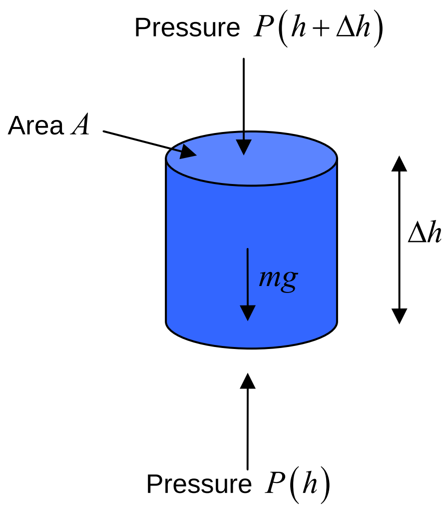
He established Boyle's Law,

$$PV = \mathbf{const}$$

for the range of pressures he used. It is important to note that in his experiments he allowed a long enough time between volume measurements for the trapped air to get back to room temperature.

The Law of Atmospheres: An Ocean of Water

First, a quick reminder of how we found the pressure variation with depth in an ocean of water at rest. We imagine isolating a small cylinder of water, with its axis vertical, and construct a free body diagram:



The pressure forces from the surrounding water acting on the *curved* sides obviously all cancel each other. So the only forces that count are the weight of the cylinder of water, and the pressure forces on the top and the bottom—that on the bottom being greater, since it must balance the pressure on the top plus the weight, since the cylinder is at rest.

Taking the cylinder to have cross-section area A, height Δh, and the water to have density ρ, the cylinder has volume $A\Delta h$, mass $\rho A\Delta h$, and therefore weight $\rho A\Delta hg$.

The pressure P is a function of height h above the bottom, $P = P(h)$.

We've measured h here from the *bottom* of the ocean, because in the next section, we'll apply the same analysis to the atmosphere, where we *do* live at the bottom of the "sea".

The pressure on top of the cylinder exerts a downward force equal to

$$\text{pressure} \times \text{area} = P(h+\Delta h)A$$

the bottom feels an *upward* pressure $P(h)A$, so since the total force must be zero,

$$P(h+\Delta h)A - P(h)A + \rho A\Delta hg = 0.$$

This equation can be rearranged to:

$$\frac{P(h+\Delta h)-P(h)}{\Delta h} = -\rho g.$$

Recalling that the differential is defined by $\dfrac{df(x)}{dx} = \lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$, we see that this pressure equation in the limit $\Delta h \to 0$ becomes:

$$dP(h)/dh = -\rho g\ .$$

Since ρg is a constant, the solution is simple:

$$P(h) = -\rho g(h - h_0)$$

where we've written the constant of integration in the form $\rho g h_0$. Notice the pressure in this ocean drops to zero at height $h = h_0$ – obviously the surface! This means our formula describes water pressure in an ocean of depth h_0, and is just a different way of writing that the pressure is ρg times the depth below the surface. (We are subtracting off the atmospheric pressure acting down on the ocean's surface from the air above it—we're just considering the extra pressure from the weight of the water itself as we descend. Remember air pressure is the same as approximately thirty feet of water, so is a small correction in a real ocean)

An Ocean of Air

We now go through exactly the same argument for an "ocean of air", drawing the same free body diagram for a small vertical cylinder, and arriving at the same differential equation,

$$dP(h)/dh = -\rho g\ .$$

But it doesn't have the same solution! The reason is that ρ, which we took to be constant for water (an excellent approximation), is obviously *not* constant for air. It is well known that the air thins out with increasing altitude.

The key to solving this equation is Boyle's Law: for a given quantity of gas, it has the form $PV = \text{const.}$, but notice that means that if the pressure of the gas is doubled, the gas is compressed into half the space, so *its density is also doubled*.

So an alternative way to state Boyles law is

$$\rho(h) = CP(h)$$

where C is a constant (assuming constant temperature). Putting this in the differential equation:

$$dP(h)/dh = -CP(h)g\ .$$

This equation can be solved (if this is news to you, see the footnote at end of this section):

$$P(h) = P_0 e^{-Cgh}.$$

The air density decreases *exponentially* with height: this equation is the **Law of Atmospheres**.

This density decrease doesn't happen with water because water is practically incompressible. One analogy is to imagine the water to be like a tower of bricks, one on top of the other, and the air a tower of brick-shaped sponges, so the sponges at the bottom are squashed into much greater density—but this isn't quite accurate, because at the top of the atmosphere, the air gets thinner and thinner without limit, unlike the sponges.

Footnote: Solving the Differential Equation

The equation is the same as $\frac{df(x)}{dx} = af(x)$, where a is a constant. If you are already familiar with the exponential function, and know that $\frac{d}{dx}e^{ax} = ae^{ax}$, you can see the equation is solved by the exponential function. Otherwise, the equation can be rearranged to $\frac{df}{f} = adx$, then integrated using $\int \frac{df}{f} = \ln f$ to give $\ln f(x) = ax + c$, with c a constant of integration. Finally, taking the exponential of each side, using $e^{\ln f(x)} = f(x)$, gives $f(x) = Ce^{ax}$, where $C = e^c$.

Exercises

1. Atmospheric pressure varies from day to day, but **1 atm** is defined as 1.01 x 10^5 Pa. Calculate how far upwards such a pressure would force a column of water in a "water barometer".

2. The density of air at room temperature is about 1.29 kg/m^3. Use this together with the definition of 1 atm above to find the constant C in the Law of Atmospheres written above. Use your result to estimate the atmospheric pressure on top of the Blue Ridge (say 4000 feet), Snowmass (11,000 feet) and Mount Everest (29,000 feet).

3. As a practical matter, how would you measure the density of air in a room? Actually, Galileo did this in the early 1600's. Can you figure out how he managed to do it? (His result was off by a factor of two, but that was still pretty good!)

The Bernoulli Effect

Contrary to most peoples' intuition, when fluid flowing through a pipe encounters a narrower section, the pressure in the fluid goes down. We show how this must follow from Newton's Laws, and demonstrate the effect.

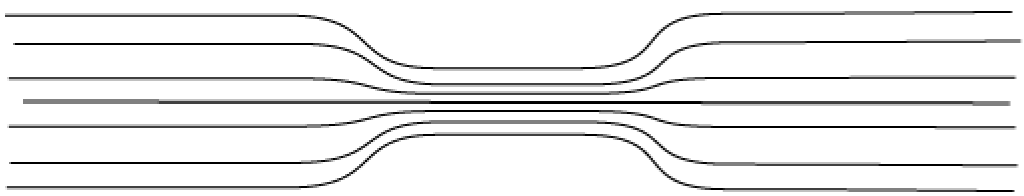
Suppose air is being pumped down a smooth round tube, which has a constant diameter except for a section in the middle where the tube narrows down to half the diameter, then widens out again. Assume all the changes in diameter take place smoothly, and the air flows steadily down the tube, with no eddies or turbulence.

Question: where in the tube do you expect the pressure to be greatest?

Most people asked this for the first time predict that the pressure will be greatest in the narrow portion of the tube. But in fact, if we actually do the experiment, by putting pressure gauges at various points along the tube, we find, counter intuitively, that *the air pressure is lowest where the air is moving fastest*!

The difference in heights of the dark liquid in the two arms of the U-tubes measures the pressure difference between that point in the flow tube and the outside atmospheric pressure.

To see how this could be, we will apply the techniques we developed to find how pressure varied in a *stationary* fluid. The way we did that, remember, was by drawing a free body diagram for a small cylinder of fluid. Since this small cylinder was at rest, the total force on it was zero, so the net pressure balanced the weight. Now consider a steadily *moving* fluid. It's helpful to visualize the flow by drawing in streamlines, lines such that their direction is the direction the fluid is moving in at each point.



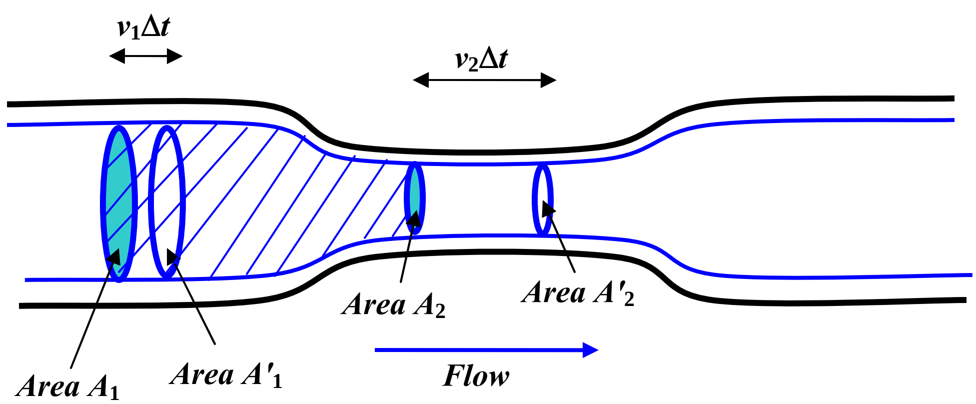
Actually, these streamlines not only tell you the *direction* the fluid is moving in, but also gives some idea of the *speed*—where they come closer together, the fluid must be moving faster, because the same amount of fluid is flowing through a narrower region.

Imagine now a cylinder of air moving along the pipe, its axis parallel to the streamline. Obviously, it must speed up as it enters the narrow part of the tube—since the same amount of air is flowing through the narrow part as the wide part, it must be going faster.

But if the small cylinder of fluid is accelerating, it must be acted on by a force pushing it from behind.

Its weight is irrelevant here, since it's moving horizontally. Therefore the only force acting on it is the pressure, and we have to conclude that the pressure at its back is greater than the pressure on its front. *Therefore the pressure must be dropping on entering the narrow part*.

To make clearer what's going on, we'll draw a rather large cylinder:



The fluid is flowing steadily and smoothly along the pipe. The thick blue lines are streamlines, in fact you should imagine rotating the whole diagram about the central axis to get a three-dimensional picture, and the blue lines would become a cylinder, with a narrower "neck" section, echoing the shape of the pipe.

Now consider the body of fluid within the streamlines shown, and capped at the two ends by the circular areas A_1 and A_2. The rate of flow of fluid across A_1 must be the same as the rate of flow across A_2, because in steady flow fluid can't be piling up in the middle (or depleting from there either). The volume flowing across A_1 in one second is v_1A_1. (To see this, imagine a long straight pipe without a narrow part. If the fluid is flowing at, say 3 meters per second, then in one second all the fluid which was within 3 meters of the area A_1 on the upstream side will have flowed through.)

So, flow across A_1 equals flow across A_2,

$$A_1 v_1 = A_2 v_2 .$$

(*Footnote*: It's perhaps worth mentioning that we are implicitly assuming the velocity is the same at all points on area A_1. Any real fluid has some viscosity (friction) and will be moving more slowly near the sides of the pipe than in the middle. We'll discuss this later. For now, we consider an "ideal" fluid, the term used when one ignores viscosity. In fact, the result we derive is ok—we could have taken a tiny area A_1 far away from the sides, so that the velocity would have been the same for the whole area, but that would have given a much less clear diagram.)

We're now ready to examine the increase in kinetic energy of the fluid as it speeds up into the narrow part, and understand how the pressure difference did the work necessary to speed it up.

Suppose that after a time Δt, the fluid that was at an initial instant between A_1 and A_2 has moved to the volume between A'_1 and A'_2. As far as the chunk of fluid we're tracking is concerned, it has effectively replaced a volume $A_1 v_1 \Delta t$ moving at v_1 with a volume $A_2 v_2 \Delta t$ moving at v_2. But remember $A_1 v_1 = A_2 v_2$, so if the density of the fluid is ρ, we're talking about a mass of fluid $\rho A_1 v_1 \Delta t$ which has effectively increased in speed from v_1 to v_2. That is to say, the increase in kinetic energy is just

$$\Delta\left(K.E.\right) = \tfrac{1}{2}\left(\rho A_1 v_1 \Delta t\right)\left(v_2^2 - v_1^2\right).$$

The only possible source for this increase in energy is the work done by pressure in pushing the fluid into the narrow part.

Taking the pressure on area A_1 to be P_1, the total force on A_1 is $P_1 A_1$. In the time Δt, this force acts through a distance $v_1 \Delta t$, and hence does work $=$ force $\times$ distance $= P_1 A_1 v_1 \Delta t$.

So this is work done on our chunk of fluid by the fluid pushing it from behind—but that's not the end of the story, because our chunk of fluid *itself* does work pushing the fluid in front of it, so to find the *total* increase in our chunk's energy, we must subtract off the external work it does. That is, the total work done by pressure on our fluid is

$$P_1 A_1 v_1 \Delta t - P_2 A_2 v_2 \Delta t = \left(P_1 - P_2\right) A_1 v_1 \Delta t$$

remembering that $A_1 v_1 = A_2 v_2$.

This work done must equal the change in kinetic energy, so

$$\left(P_1 - P_2\right) A_1 v_1 \Delta t = \tfrac{1}{2}\left(\rho A_1 v_1 \Delta t\right)\left(v_2^2 - v_1^2\right)$$

from which

$$P_1 + \tfrac{1}{2}\rho v_1^2 = P_2 + \tfrac{1}{2}\rho v_2^2.$$

This is *Bernoulli's equation*.

There is a further easy generalization: we could have the pipe sloping uphill. In that case, the fluid would gain potential energy as well as kinetic energy, so the pressure would have to do more work. If we take the center of the area A_1 to be at height h_1, the area A_2 at h_2, and take Δt very small, the increase in potential energy in time Δt will be $(\rho A_1 v_1 \Delta t) g (h_2 - h_1)$, and Bernoulli's equation becomes:

$$P_1 + \tfrac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \tfrac{1}{2}\rho v_2^2 + \rho g h_2.$$

Viscosity

After briefly reviewing friction between solids, we examine viscosity in liquids and gases, building up some understanding of what's going on at the molecular level. This makes it possible to understand some surprising results: for example, the viscosity of a gas does not change if the gas is compressed to greater density.

Introduction: Friction at the Molecular Level

Viscosity is, essentially, fluid friction. Like friction between moving solids, viscosity transforms kinetic energy of (macroscopic) motion into heat energy. Heat is energy of random motion at the molecular level, so to have any understanding of how this energy transfer takes place, it is essential to have some picture, however crude, of solids and/or liquids sliding past each other as seen on the molecular scale.

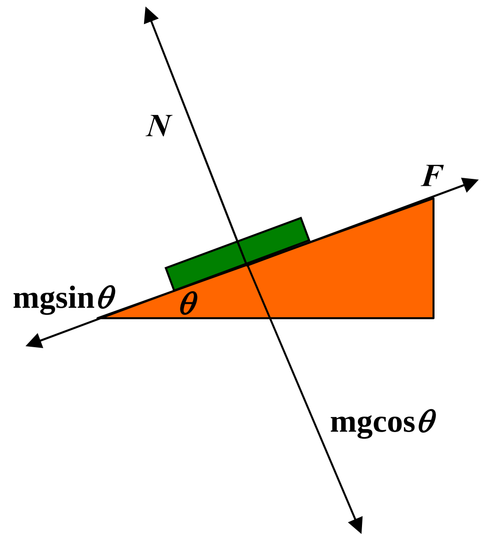
To begin with, we'll review the molecular picture of friction between *solid* surfaces, and the significance of the coefficient of friction μ in the familiar equation $F = \mu N$. Going on to fluids, we'll give the definition of the coefficient of viscosity for liquids and gases, give some values for different fluids and temperatures, and demonstrate how the microscopic picture can give at least a qualitative understanding of how these values vary: for example, on raising the temperature, the viscosity of liquids *decreases*, that of gases *increases*. Also, the viscosity of a gas doesn't depend in its density! These mysteries can only be unraveled at the molecular level, but there the explanations turn out to be quite simple.

As will become clear later, the coefficient of viscosity η can be viewed in two rather different (but of course consistent) ways: it is a measure of how much heat is generated when faster fluid is flowing by slower fluid, but it is also a measure of the *rate of transfer of momentum* from the faster stream to the slower stream. Looked at in this second way, it is analogous to thermal conductivity, which is a measure of the rate of transfer of heat from a warm place to a cooler place.

Quick Review of Friction Between Solids

First, *static* friction: suppose a book is lying on your desk, and you tilt the desk. At a certain angle of tilt, the book begins to slide. Before that, it's held in place by "static friction". What does that mean on a molecular level? There must be some sort of attractive force between the book and the desk to hold the book from sliding.

Let's look at all the forces on the book: gravity is pulling it vertically down, and there is a "normal force" of the desk surface pushing the book in the direction normal to the desk surface. (This normal force is the springiness of the desktop, slightly compressed by the weight of the book.) When the desk is tilted, it's best to visualize the vertical gravitational force as made up of a component normal to the surface and one parallel to the surface (downhill). The gravitational component perpendicular to the surface is exactly balanced by the normal force, and if the book is at rest, the "downhill" component of gravity is balanced by a frictional force parallel to the surface in the uphill direction. On a microscopic scale, this static frictional force is from fairly short range attractions between molecules on the desk and those of the book.



Question: but if that's true, why does doubling the normal force double this frictional force? (Recall $F = \mu N$, where N is the normal force, F is the limiting frictional force just before the book begins to slide, and μ is the coefficient of friction. By the way, the first appearance of F being proportional to N is in the notebooks of Leonardo da Vinci.)

Answer : Solids are almost always *rough* on an atomic scale: when two solid surfaces are brought into contact, in fact only a tiny fraction of the common surface is really in contact at the atomic level. The stresses within that tiny area are large, the materials distort plastically and there is adhesion. The picture can be very complex, depending on the materials involved, but the bottom line is that there is only atom-atom interaction between the solids *over a small area*, and what happens in this small area determines the frictional force. If the normal force is doubled (by adding another book, say) the tiny area of contact between the bottom book and the desk will also double—the true area of atomic contact *increases linearly with the normal force*—that's why friction is proportional to N. Within the area of "true contact" extra pressure makes little difference. (Incidentally, if two surfaces which really *are* flat at the atomic level are put together, there is bonding. This can be a real challenge in the optical telecommunications industry, where wavelength filters (called etalons) are manufactured by having extremely flat, highly parallel surfaces of transparent material separated by distances comparable to the wavelength of light. If they touch, the etalon is ruined.)

On tilting the desk more, the static frictional force turns out to have a limit—the book begins to slide. But there's still some friction: experimentally, the book does not have the full acceleration the component of gravity parallel to the desktop should deliver. This must be because in the area of contact with the desk the two sets of atoms are constantly colliding, loose bonds are forming and breaking, some atoms or molecules fall away. This all causes a lot of atomic and molecular vibration at the surface. In other words, some of the gravitational potential energy the sliding book is losing is ending up as heat instead of adding to the book's kinetic energy. This is the familiar dynamic friction you use to warm your hands by rubbing them together in wintertime. It's often called *kinetic* friction. Like static friction, it's proportional to the normal force: $F = \mu_K N$. The proportionality to the normal force is for the same reason as in the static case: the kinetic frictional drag force also comes from the tiny area of true atomic contact, and this *area* is proportional to the normal force.

A full account of the physics of friction (known as tribology) can be found, for example, in *Friction and Wear of Materials*, by Ernest Rabinowicz, second Edition, Wiley, 1995.

Liquid Friction

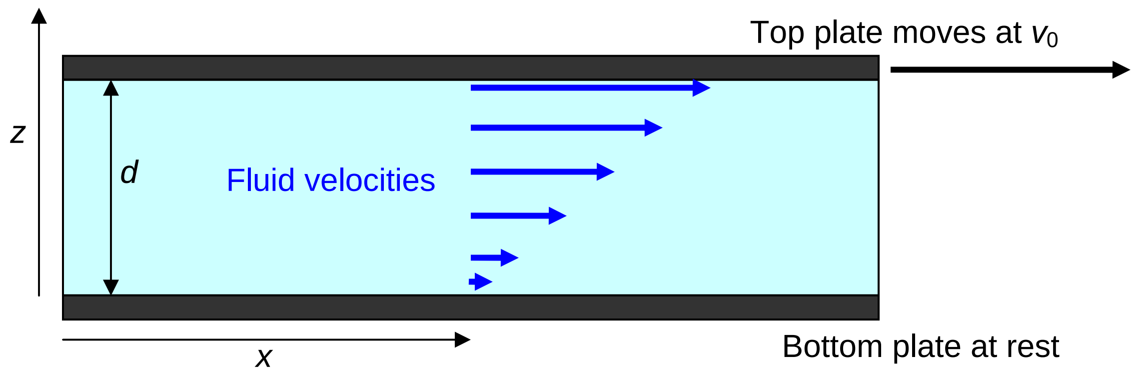
What happens if instead of two solid surfaces in contact, we have a solid in contact with a liquid? First, there's no such thing as static friction between a solid and a liquid. If a boat is at rest in still water, it will move in response to the slightest force. Obviously, a tiny force will give a tiny acceleration, but that's quite different from the book on the desk, where a considerable force gave no acceleration at all. But there *is* dynamic liquid friction—even though an axle turns a lot more easily if oil is supplied, there is still *some* resistance, the oil gets warmer as the axle turns, so work is being expended to produce heat, just as for a solid sliding across another solid.

One might think that for solid/liquid friction there would be some equation analogous to $F = \mu_K N$: perhaps the liquid frictional force is, like the solid, proportional to pressure? But experimentally this turns out to be false—there is little dependence on pressure over a very wide range. The reason is evidently that since the liquid can flow, there is good contact over the whole common area, even for low pressures, in contrast to the solid/solid case.

Newton's Analysis of Viscous Drag

Isaac Newton was the first to attempt a quantitative definition of a coefficient of viscosity. To make things as simple as possible, he attempted an experiment in which the fluid in question was sandwiched between two large parallel horizontal plates. The bottom plate was held fixed, the top plate moved at a steady speed v_0, and the drag force from the fluid was measured for different values of v_0, and different plate spacing. (Actually Newton's experiment didn't work too well, but as usual his theoretical reasoning was fine, and fully confirmed experimentally by Poiseuille in 1849 using liquid flow in tubes.)

Newton assumed (and it has been amply confirmed by experiment) that at least for low speeds the fluid settles into the flow pattern shown below. The fluid in close contact with the bottom plate stays at rest, the fluid touching the top plate gains the same speed v_0 as that plate, and in the space between the plates the speed of the fluid increases linearly with height, so that, for example, the fluid halfway between the plates is moving at ½ v_0:



Just as for kinetic friction between solids, to keep the top plate moving requires a steady force. Obviously, the force is proportional to the total amount of fluid being kept in motion, that is, to the total area of the top plate in contact with the fluid. The significant parameter is the *horizontal force per unit area of plate*, F/A, say. This clearly has the same dimensions as pressure (and so can be measured in Pascals) although it is physically completely different, since in the present case the force is *parallel* to the area (or rather to a line within it), not perpendicular to it as pressure is.

(*Note for experts only*: Actually, viscous drag and pressure are not completely unrelated—as we shall discuss later, the viscous force may be interpreted as a rate of transfer of momentum into the fluid, momentum parallel to the surface that is, and pressure can *also* be interpreted as a rate of transfer of momentum, but now perpendicular to the surface, as the molecules bounce off. Physically, the big difference is of course that the pressure doesn't have to do any work to keep transferring momentum, the viscous force does.)

Newton conjectured that the necessary force F/A would be proportional to the velocity gradient in the vicinity of the top plate. In the simple geometry above, the velocity gradient is the same everywhere between the plates, v_0/d, so

$$F / A = \eta v_0 / d$$

defines the *coefficient of viscosity* η. The SI units of η are **Pascal.seconds**, or **Pa.s**.

A convenient unit is the milliPascal.second, mPa.s. (It happens to be close to the viscosity of water at room temperature.) Confusingly, there is another set of units out there, the poise, named after Poiseuille—usually seen as the centipoise, which happens to equal the millipascal.second! And, there's *another* viscosity coefficient in common use: the kinetic viscosity, $\nu = \mu / \rho$, where ρ is the fluid density. This is the relevant parameter for fluids flowing downwards gravitationally. But we'll almost always stick with η.

Here are some values of η for common liquids:

Liquid	*Viscosity in mPa.s*
Water at 0°C	*1.79*
Water at 20°C	*1.002*
Water at 100°C	*0.28*
Glycerin at 0°C	*12070*
Glycerin at 20°C	*1410*
Glycerin at 30°C	*612*
Glycerin at 100°C	*14.8*
Mercury at 20°C	*1.55*
Mercury at 100°C	*1.27*
Motor Oil SAE 30	*200*
Motor Oil SAE 60	*1000*
Ketchup	*50,000*

Some of these are obviously ballpark – the others probably shouldn't be trusted to be better that 1% or so, glycerin maybe even 5-10% (see CRC Tables); these are quite difficult measurements, very sensitive to purity (glycerin is hygroscopic) and to small temperature variations.

To gain some insight into these very different viscosity coefficients, we'll try to analyze what's going on at the molecular level.

A Microscopic Picture of Viscosity in Laminar Flow

For Newton's picture of a fluid sandwiched between two parallel plates, the bottom one at rest and the top one moving at steady speed, the fluid can be pictured as made up of many layers, like a pile of printer paper, each sheet moving a little faster than the sheet below it in the pile, the top sheet of fluid moving with the plate, the bottom sheet at rest. This is called *laminar flow*: laminar just means sheet (as in laminate, when a sheet of something is glued to a sheet of something else). If the top plate is gradually speeded up, at some point laminar flow becomes unstable and turbulence begins. We'll assume here that we're well below that speed.

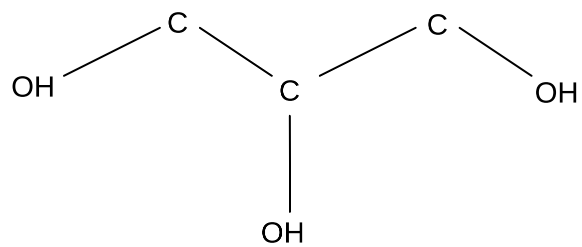
So where's the friction? It's not between the fluid and the plates (or at least very little of it is—the molecules right next to the plates mostly stay in place) it's *between the individual sheets*—throughout the fluid. Think of two neighboring sheets, the molecules of one bumping against their neighbors as they pass. As they crowd past each other, on average the molecules in the faster stream are slowed down, and those in the slower stream speeded up. Of course, *momentum* is always conserved, but the macroscopic kinetic energy of the sheets of fluid is partially lost—transformed into heat energy.

Exercise: Suppose a mass m of fluid moving at v_1 in the x-direction mixes with a mass m moving at v_2 in the x-direction. Momentum conservation tells us that the mixed mass $2m$ moves at $\frac{1}{2}(v_1 + v_2)$. Prove that the total kinetic energy has *decreased* if v_1, v_2 are unequal.

This is the fraction of the kinetic energy that has disappeared into heat.

This molecular picture of sheets of fluids moving past each other gives some insight into why viscosity decreases with temperature, and at such different rates for different fluids. As the molecules of the faster sheet jostle past those in the slower sheet, remember they are all jiggling about with thermal energy. The jiggling helps break them loose if they get jammed temporarily against each other, so as the temperature increases, the molecules jiggle more furiously, unjam more quickly, and the fluid moves more easily—the viscosity goes down.

This drop in viscosity with temperature is dramatic for glycerin. A glance at the molecule suggests that the zigzaggy shape might cause jamming, but the main cause of the stickiness is that the outlying H's in the OH groups readily form hydrogen bonds (see *Atkins' Molecules*, Cambridge).



Glycerin Molecule: other carbon valence bonds have H atoms.

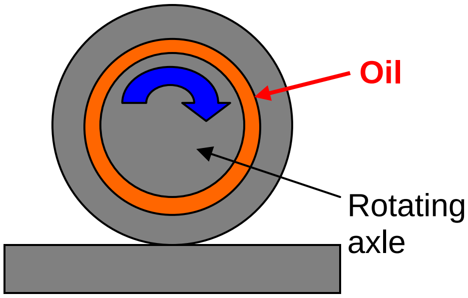
For mercury, a fluid of round atoms, the drop in viscosity with temperature is small. Mercury atoms don't jam much, they mainly just bounce off each other (but even that bouncing randomizes their direction, converting macroscopic kinetic energy to heat). Water molecules are in between glycerin and mercury in complexity. Looking at the table above, it is evident this simple picture makes at least qualitative sense of the data.

Another mechanism generating viscosity is the diffusion of faster molecules into the slower stream and vice versa. As discussed below, this is far the dominant factor in viscosity of gases, but is *much* less important in liquids, where the molecules are crowded together and constantly bumping against each other.

This temperature dependence of viscosity is a real problem in lubricating engines that must run well over a wide temperature range. If the oil gets too runny (that is, low viscosity) it will not keep the metal surfaces from grinding against each other; if it gets too thick, more energy will be needed to turn the axle. "Viscostatic" oils have been developed: the natural decrease of viscosity with temperature ("thinning") is counterbalanced by adding polymers, long chain molecules at high temperatures that curl up into balls at low temperatures.

Oiling a Wheel Axle

The simple linear velocity profile pictured above is actually a good model for ordinary lubrication. Imagine an axle of a few centimeters diameter, say about the size of a fist, rotating in a bearing, with a 1 mm gap filled with SAE 30 oil, having η = 200 mPa.s. (Note: mPa, millipascals, not Pascals! 1Pa = 1000mPa.)



If the total cylindrical area is, say, 100 sq cm., and the speed is 1 m.s^{-1}, the force per unit area (sq. m.)

$$F/A = \eta v_0/d = 200\cdot 10^{-3}\cdot 1/10^{-3} = 200$$

So for our 100 sq.cm bearing the force needed to overcome the viscous "friction" is 2*N*. At the speed of 1 m sec^{-1}, this means work is being done at a rate of 2 joules per sec., or 2 watts, which is heating up the oil. (This heat must be conducted away, or the oil continuously changed by pumping, otherwise it will get too hot.)

*Viscosity: Kinetic Energy Loss and Momentum Transfer

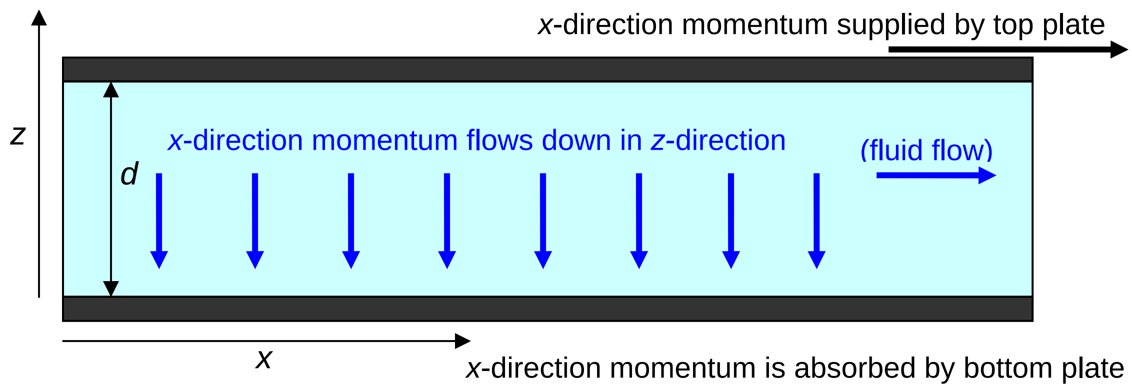
So far, we've viewed the viscosity coefficient η as a measure of friction, of the dissipation into heat of the energy supplied to the fluid by the moving top plate. But η is also the key to understanding what happens to the *momentum* the plate supplies to the fluid.

For the picture above of the steady fluid flow between two parallel plates, the bottom plate at rest and the top one moving, a steady force per unit area F/A in the *x*-direction applied to the top plate is needed to maintain the flow.

From Newton's law $F = dp/dt$, *F/A is the rate at which momentum in the x-direction is being supplied (per unit area) to the fluid*. Microscopically, molecules in the immediate vicinity of the plate either adhere to it or keep bouncing against it, picking up momentum to keep moving with the plate (these molecules also constantly *lose* momentum by bouncing off other molecules a little further away from the plate).

Question: But doesn't the total momentum of the fluid stay the same in steady flow? Where does the momentum fed in by the moving top plate go?

Answer: the *x*-direction momentum supplied at the top passes downwards from one layer to the next, ending up at the bottom plate (and everything it's attached to). Remember that, unlike kinetic energy, momentum is always conserved—it can't disappear.



So, there is a steady flow *in the z-direction* of *x*-direction momentum. Furthermore, the left-hand side of the equation

$$F/A = \eta v_0/d$$

is just this momentum flow rate. The right hand side is the coefficient of viscosity multiplied by the gradient in the *z*-direction of the *x*-direction velocity.

Viewed in this way, $F/A = \eta v_0/d$ is a *transport equation*. It tells us that the rate of transport of *x*-direction momentum downwards is proportional to the rate of change of *x*-direction velocity in that direction, and the constant of proportionality is the coefficient of viscosity. And, we can express this slightly differently by noting that the rate of change of *x*-direction *velocity* is proportional to the rate of change of *x*-direction *momentum density*.

Recall that we mentioned earlier the so-called *kinetic* viscosity coefficient, $\nu = \mu/\rho$. Using that in the equation

$$F/A = \eta v_0/d = \nu\rho v_0/d\ ,$$

replaces the velocity gradient with a *x*-direction *momentum* gradient. To abbreviate a clumsy phrase, let's call the *x*-direction momentum density π_x, and the current of this in the *z*-direction $J_z(\pi_x)$. Then our equation becomes

$$J_z(\pi_x) = \nu\frac{d\pi_x}{dz}.$$

The current of π_x in the *z*-direction is proportional to how fast π_x is changing in that direction.

This closely resembles heat flowing from a hot spot to a cold spot: heat energy flows towards the place where there is less of it, "downhill" in temperature. The rate at which it flows is proportional to the temperature gradient, and the constant of proportionality is the thermal conductivity (see later). Here, the π_x momentum flow is analogous: it too flows to where there is less of it, and the kinetic viscosity coefficient corresponds to the thermal conductivity.

*Viscosity in Gases

Suppose now we repeat Newton's suggested experiment, the two parallel plates with one at rest the other moving at steady speed, but with *gas* rather than liquid between the plates.

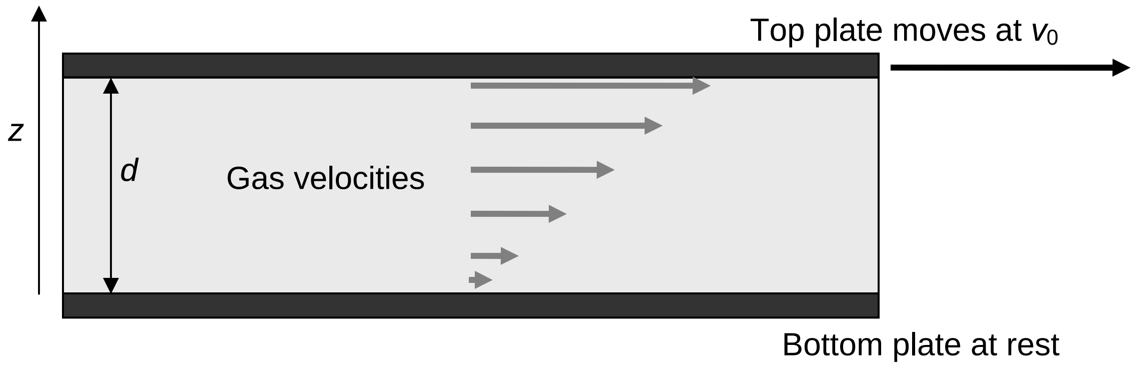
It is found experimentally that the equation $F / A = \eta v / d$ still describes the force necessary to maintain steady motion, but, not surprisingly, for gases anywhere near atmospheric pressure the coefficient of viscosity is far lower than that for liquids (not counting liquid helium—a special case):

Gas	*Viscosity in 10^{-6} Pa.s*
Air at 100K	*7.1*
Air at 300K	*18.6*
Air at 400K	*23.1*
Hydrogen at 300K	*9.0*
Helium at 300K	*20.0*
Oxygen at 300K	*20.8*
Nitrogen at 300K	*17.9*
Xenon at 300K	*23.2*

These values are from the CRC Handbook, 85th Edition, **6**-201.

Note first that, in contrast to the liquid case, gas viscosity *increases* with temperature. Even more surprising, it is found experimentally that over a very wide range of densities, gas viscosity is *independent of the density* of the gas!

Returning to the two plates, and picturing the gas between as made up of layers moving at different speeds as before:



The first thing to realize is that at atmospheric pressure the molecules take up something like *a thousandth* of the volume of the gas. The previous picture of crowded molecules jostling each other is completely irrelevant! As we shall discuss in more detail later, the molecules of air at room temperature fly around at about 500 meters per second, the molecules have diameter around 0.35 nm, are around 3 or 4 nm apart on average, and travel of order 70 nm between collisions with other molecules.

So where does the gas viscosity come from? Think of two adjacent layers of gas moving at different speeds. Molecules from the fast layer fly into the slow layer, where after a collision or two they are slowed to go along with the rest. At the same time, some slower molecules fly into the fast layer. Even if we assume that kinetic energy is conserved in each individual molecular collision (so we're ignoring for the moment excitation of internal modes of the molecules) the *macroscopic* kinetic energy of the layers of gas decreases overall (see the exercise in the preceding section). How can that be? Isn't energy conserved? Yes, total energy *is* conserved, what happens is that some of the macroscopic kinetic energy of the gas moving as a smooth substance has been transferred into the microscopic kinetic energy of the individual molecules moving in random directions within the gas, in other words, into the random molecular kinetic energy we call heat.

*Estimating the Coefficient of Viscosity for a Gas: Momentum Transfer and Mean Free Path

The way to find the viscosity of a gas is to calculate the rate of z-direction (downward) transfer of x-momentum, as explained in the previous section but one.

The moving top plate maintains a steady horizontal flow pattern, the x-direction speed at height z

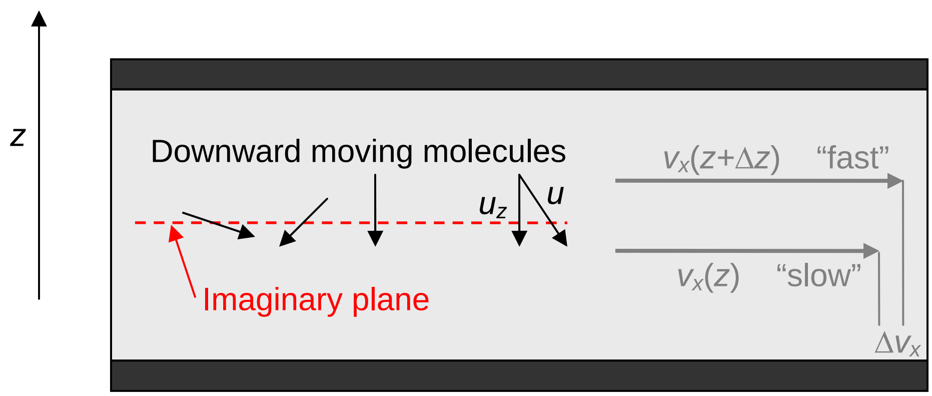
$$v(z) = v_0 z / d$$

As explained above, the moving plate is feeding x-direction momentum into the gas at a rate $F / A = \eta v_0 / d$, this momentum moves down through the gas at a steady rate, and the coefficient of viscosity tells us what this rate of momentum flow is for a given velocity gradient.

In fact, this rate of momentum flow can be calculated from a simple kinetic picture of the gas: remember the molecules have about a thousand times more room than they do in the liquid state, so the molecules go (relatively) a long way between collisions. We shall examine this kinetic picture of a gas in much more detail later in the course, but for now we'll make the simplifying assumption that the molecules all have speed u, and travel a distance l between collisions. Actually this approximation gets us pretty close to the truth.

We take the density of molecules to be n, the molecular mass m. To begin thinking about x-direction momentum moving downwards, imagine some plane parallel to the plates and between them. Molecules from above are shooting through this plane and colliding with molecules in the slower moving gas below, on average transferring a little extra momentum in the x-direction to the slower stream. At the same time, some molecules from the slower stream are shooting upwards and will slow down the faster stream.

Let's consider first the molecules passing through the imaginary plane from above: we're only interested in the molecules already moving *downwards*, that's half of them, so a molecular density of $n/2$. If we assume for simplicity that all the molecules move at the same speed u, then the average *downward* speed of these molecules $\overline{u_z} = u/2$ ($u_z = u\cos\theta$, and the average value of $\cos\theta$ over all downward pointing directions is ½.)



Thus the number of molecules per second passing through the plane from above is

$$\frac{n}{2}\overline{|u_z|} = \frac{nu}{4}$$

and the same number are of course coming *up* from below! (Not shown.) (To see what's going on, the mean free path shown here is hugely exaggerated compared with the distance between the plates, of course.)

But this isn't quite what we want: we need to know how *efficiently* these molecules crossing the plane are transferring momentum for the fast moving streams above to the slower ones below. Consider one particular molecule going from the faster stream at some downward angle into the slower stream.

Let us assume it travels a distance l from its last collision in the "fast" stream to its first in the "slow" stream. The average distance between collisions is called the **mean free path**, here "mean" is used in the sense of "average", and is denoted by l. We are simplifying slightly by taking all distances between collisions to be l, so we don't bother with statistical averaging of the distance traveled. This does not make a big difference. The distance traveled *in the downward direction* is $\Delta z = lu_z / u$, so the (x-direction) speed difference between the two streams is

$$\Delta v = \frac{dv(z)}{dz}\Delta z = \frac{v_0}{d}\cdot\frac{lu_z}{u}.$$

The molecule has mass m, so *on average* the momentum transferred from the fast stream to the slow stream is $\Delta p = m\Delta v$. With our simplifying assumption that all molecules have the same speed u, all downward values of u_z between 0 and u are equally likely, and the density of downward-moving molecules is $n/2$, so the rate of transfer of momentum by downward-moving molecules through the plane is

$$\frac{n}{2}\overline{u_z m\Delta v} = \frac{nm}{2}\cdot l\cdot\frac{\overline{u_z^2}}{u}\cdot\frac{v_0}{d}.$$

At the same time, molecules are moving upwards form the slower streams into the faster ones, and the calculation is exactly the same. These two processes have the same sign: in the first case, the slower stream is gaining forward momentum from the faster, in the second, the faster stream is losing forward momentum, and in both cases total forward momentum is conserved. Therefore, the two processes make the same contribution, and the total momentum flow rate (per unit area) across the plane is

$$\text{momentum transfer rate} \;=\; nm\cdot l\cdot\frac{\overline{u_z^2}}{u}\cdot\frac{v_0}{d} = \frac{1}{3}nmlu\cdot\frac{v_0}{d}$$

using $u_z^2 = u^2/3$.

Evidently this rate of downward transfer of x-direction momentum doesn't depend on what level between the plates we choose for our imaginary plane, it's the same momentum flow all the way from the top plate to the bottom plate: so it's simply the rate at which the moving top plate is supplying x-direction momentum to the fluid,

$$\text{momentum supply rate} \;= F/A = \eta v_0 / d.$$

Since the momentum supplied moves steadily downwards through the fluid, the supply rate *is* the transfer rate, the two equations above are for the same thing, and we deduce that the coefficient of viscosity

$$\eta = \frac{1}{3} nmlu.$$

In deriving this formula, we *did* make the simplifying assumption that all the molecules move at the same speed, but in fact the result is very close to correct for the more general case.

*Why Doesn't the Viscosity of a Gas Depend on Density?

Imagine we have a gas made up of equal numbers of red and green molecules, which have the same size, mass, etc. One of the molecules traveling through will on average have half its collisions with red molecules, half with green. If now all the red molecules suddenly disappear, the collision rate for our wandering molecule will be halved. This means its mean free path l will double. So, since the coefficient of viscosity $\eta = \frac{1}{3} nmlu$, halving the gas number density n at the same time doubles the mean free path l, so η *is unchanged*. (η does finally begin to drop when there is so little gas left that the mean free path is of order the size of the container.)

Another way to see this is to think about the molecules shooting down from the faster stream into slower streams in the two-plate scenario. If the density is halved, there will only be half the molecules moving down, but each will deliver the x-momentum difference twice as far—the further they go, the bigger the x-velocity difference between where they begin and where they end, and the more effective they are in transporting x-momentum downwards.

*Comparing the Viscosity Formula with Experiment

We'll repeat the earlier table here for convenience:

Gas	*Viscosity in 10^{-6} Pa.s*
Air at 100K	*7.1*
Air at 300K	*18.6*
Air at 400K	*23.1*
Hydrogen at 300K	*9.0*
Helium at 300K	*20.0*
Oxygen at 300K	*20.8*
Nitrogen at 300K	*17.9*
Xenon at 300K	*23.2*

These values are from the CRC Handbook, 85th Edition, **6**-201.

It's easy to see one reason why the viscosity increases with temperature: from $\eta = (1/3)nmlu$, η is proportional to the average molecular speed *u*, and since this depends on temperature as $\frac{1}{2}m\overline{u^2} = \frac{3}{2}k_BT$ (if this is unfamiliar, be assured we'll be discussing it in detail later), this factor contributes a $\sqrt{T}$ dependence. In fact, though, from the table above, the increase in viscosity with temperature is more rapid than $\sqrt{T}$. We know the density and mass remain constant (we're far from relativistic energies!) so if the analysis is correct, the mean free path must also be increasing with temperature. In fact this is what happens—many of the changes of molecular direction in flight are not caused by hard collisions with other molecules, but by longer range attractive forces (van der Waals forces) when one molecule simply passes reasonably close to another. Now these attractive forces obviously act for a shorter time on a faster molecule, so it is deflected less. This means that as temperature and molecular speed increase, the molecules get further in approximately the same direction, and therefore transport momentum more effectively.

Comparing viscosities of different gases, at the same temperature and pressure they will have the same number density *n*. (Recall that a mole of any gas has Avogadro's number of molecules, 6×10^{23}, and that at standard temperature and pressure this occupies 22.4 liters, for any gas, if tiny pressure and volume corrections for molecular attraction and molecular volume are ignored.) The molecules will also have the same kinetic energy since they're at the same temperature (see previous paragraph).

Comparing hydrogen with oxygen, for example, the molecular mass *m* is up by a factor 16 but the velocity *u* decreases by a factor 4 (since the molecular kinetic energies $\frac{1}{2}mv^2$ are the same at the same temperature). In fact (see the table above), the viscosity of oxygen is only *twice* that of hydrogen, so from $\eta = \frac{1}{3}nmlu$ we conclude that the hydrogen molecule has twice the mean free path distance l between collisions—not surprising since it is smaller. Also, helium must have an even longer mean free path, again not surprising for the smallest molecule in existence. The quite large difference between nitrogen and oxygen, next to each other in the periodic table, is because N_2 has a trivalent bonding, tighter than the divalent O_2 bonding, and in fact the internuclear distance in the N_2 molecule is 10% less than in O_2. Xenon is heavy (atomic weight 131), but its mean free path is shorter than the others because the atom is substantially larger, and so an easier target.

Historically, the sizes of many atoms and molecules were first estimated from viscosity measurements using this method. In fact, just such a table of atomic and molecular diameters, calculated on the assumption that the atoms or molecules are hard spheres, can be found in the CRC tables, 85$^{\text{th}}$ edition, **6**-47. But the true picture is more complicated: the atoms are not just hard spheres, as mentioned earlier they have van der Waals attractive forces between them, beyond the outermost shell. Actually, the electronic densities of atoms and molecules can now be found fairly precisely by quantum calculations using self-consistent field methods, and the resulting "radii" are in rough agreement with those deduced from viscosity (there is no obvious natural definition of radius for the electron cloud).

More General Laminar Flow Velocity Distributions

We've analyzed a particularly simple case: for the fluid between two parallel plates, the bottom plate at rest and the top moving at steady speed v_0, the fluid stream velocity increases *linearly* from zero at the bottom to v_0 at the top. For more realistic laminar flow situations, such as that away from the banks in a wide river, or flow down a pipe, the rate of velocity increase on going from the fixed boundary (river bed or pipe surface) into the fluid is no longer linear, that is, $dv(z)/dz$ is not the same everywhere.

The key to analyzing these more general laminar flow patterns is to find the forces acting on a small area of one of the "sheets". (Or a larger area of sheet if the flow is uniform in the appropriate direction.) There will be external forces such as gravity or pressure maintaining the flow, which must balance the viscous drag forces exerted by neighboring sheets if the fluid is not accelerating. The sheet-sheet drag force is equivalent to the force $F/A = \eta v_0/d$ exerted by the top "sheet" of fluid on the top plate in the previous discussion, except that now the forces are within the fluid. As we discussed earlier, the drag force on the top plate comes from molecules close to the plate bouncing off, gaining momentum, which is subsequently transferred to other molecules a little further away. For liquids, this mechanism involves distances of order a few molecular diameters, for gases a few mean free paths. In either case, the distances are tiny on a macroscopic scale. This means that the appropriate formula for the drag force on a plate, or that between one sheet of fluid on another, is

$$\frac{F}{A} = \eta \frac{dv(z)}{dz}.$$

The *rate of change of stream velocity close to the interface* determines the drag force.

We shall show in the next lecture how this formula can be used to determine the flow pattern in a river, and that in a circular pipe.

Calculating Viscous Flow: Velocity Profiles in Rivers and Pipes

We present the calculus derivation of the smooth flow patterns for a wide river and for fluid in a circular cross-section pipe, and find the total flow for given slope or pressure drop.

Introduction

In this lecture, we'll derive the velocity distribution for two examples of laminar flow. First we'll consider a wide river, by which we mean wide compared with its depth (which we take to be uniform) and we ignore the more complicated flow pattern near the banks. Our second example is smooth flow down a circular pipe. For the wide river, the water flow can be thought of as being in horizontal "sheets", so all the water at the same depth is moving at the same velocity. As mentioned in the last lecture, the flow can be pictured as like a pile of printer paper left on a sloping desk: it all slides down, assume the bottom sheet stays stuck to the desk, each other sheet moves downhill a little faster than the sheet immediately beneath it. For flow down a circular pipe, the laminar "sheets" are hollow tubes centered on the line down the middle of the pipe. The fastest flowing fluid is right at that central line. For both river and tube flow, the drag force between adjacent small elements of neighboring sheets is given by force per unit area

$$\frac{F}{A}=\eta\frac{dv(z)}{dz}$$

where now the z-direction means perpendicular to the small element of sheet.

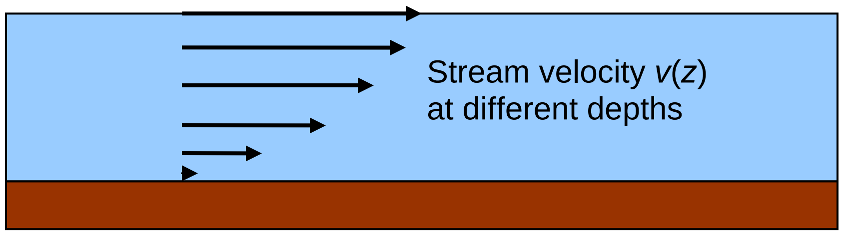
A Flowing River: Finding the Velocity Profile

For a river flowing steadily down a gentle incline under gravity, we'll assume all the streamlines point in the same direction, the river is wide and of uniform depth, and the depth is much smaller than the width. This means almost all the flow is well away from the edges (the river banks), so we'll ignore the slowing down there, and just analyze the flow rate *per meter of river width*, taking it to be uniform across the river.

The simplest basic question is: given the slope of the land and the depth of the river, what is the total flow rate?

To answer, we need to find the speed of flow $v(z)$ as a function of depth (we know the water in contact with the river bed isn't flowing at all), and then add the flow contributions from the different depths (this will be an integral) to find the total flow.

The function $v(z)$ is called the "velocity profile". We'll prove it looks something like this:



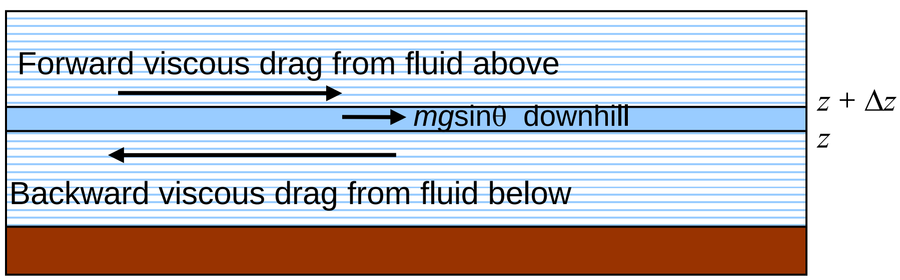
(For a smoothly flowing river, the downhill ground slope would be imperceptible on this scale.)

But how do we begin to *calculate* $v(z)$?

Recall that (in an earlier lecture) to find how hydrostatic *pressure* varied with depth, we mentally separated a cylinder of fluid from its surroundings, and applied Newton's Laws: it wasn't moving, so we figured its weight had to be balanced by the sum of the pressure forces it experienced from the rest of the fluid surrounding it. In fact, its weight was balanced by the difference between the pressure underneath and that on top.

Taking a cue from that, here we isolate mentally a thin *layer* of the river, like one of those sheets of printer paper, lying between height z above the bed and $z+\Delta z$. This layer *is* moving, but *at a steady speed*, so *the total force on it will still be zero*. Like the whole river, this layer isn't quite horizontal, its weight has a small but nonzero component dragging it downhill, and this weight component is balanced by the difference between the viscous force from the faster water above and that from slower water below.

Bear in mind that the diagram below is at a tiny angle θ to the horizontal:



Obviously, for the forces to balance, the backward drag on the thin layer from the slower moving water beneath has to be stronger than the forward drag from the faster water above, so the *rate of change* of speed with height above the river bed is *decreasing* on going up from the bed.

Let us find the total force (which must be zero) on *one square meter* of the thin layer of water between heights z and $z+\Delta z$:

First, gravity: if the river is flowing downhill at some small angle θ, this square meter of the layer (volume $\Delta z \text{ m} \times 1 \text{ m}^2 = \Delta z \text{ m}^3$, density ρ) experiences a gravitational force $mg\sin\theta \approx \rho g \Delta z \cdot \theta$ tugging it downstream (taking the small angle approximation, $\sin\theta = \theta$.)

Next, the viscous drag forces: the square meter of layer experiences *two* viscous forces, one from the slower water below, equal to $\eta dv(z)/dz$, tending to slow it down, one from the faster water above it, $\eta dv(z+\Delta z)/dz$, tending to speed it up.

Gravity must balance out the difference between the two viscous forces:

$$\rho g \theta \Delta z + \eta \frac{d}{dz} v(z+\Delta z) - \eta \frac{d}{dz} v(z) = 0$$

We can already see from this equation that, unlike the fluid between the plates, v*(*z*) can't possibly be linear in* z*—the equation* would not balance *if* dv/dz *were the same at* z *and* $z+\Delta z$ *!*

Dividing throughout by η *and by* Δz*,*

$$\frac{\frac{d}{dz} v(z+\Delta z) - \frac{d}{dz} v(z)}{\Delta z} = -\frac{\rho g \theta}{\eta}.$$

Taking now the limit $\Delta z \to 0$*, and recalling the definition of the differential*

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

we find the differential equation

$$\frac{d^2 v(z)}{dz^2} = -\frac{\rho g \theta}{\eta}.$$

The solution of this equation is easy:

$$v(z) = -\frac{\rho g \theta}{2\eta} z^2 + Cz + D$$

with *C*, *D* constants of integration.

Remember that the velocity $v(z)$ is *zero* at the bottom of the river, $z = 0$, so the constant D must be zero, and can be dropped immediately. But we're not through—we haven't found C. To do that, we need to go to the top.

Velocity Profile Near the River Surface

What happens to the thin layer of river water *at the very top*—the layer in contact with the air? Assuming there is negligible wind, there is essentially zero parallel-to-the-surface force from above.

So the balance of forces equation for the top layer is just

$$\rho g\theta\Delta z - \eta \frac{d}{dz} v(z) = 0 .$$

We can take this top layer to be as thin as we like, so let's look what happens in the limit of extreme thinness, $\Delta z \to 0$. The term $\rho g\theta\Delta z$ then goes to zero, so the other term must as well. Since η is constant, this means

$$\frac{d}{dz} v(z) = 0 \text{ at the surface } z = h.$$

So the velocity profile function $v(z)$ has zero slope at the river surface.

With this new information, we can finally fix the arbitrary integration constant C.

Now the velocity profile

$$v(z) = -\frac{\rho g\theta}{2\eta} z^2 + Cz ,$$

so

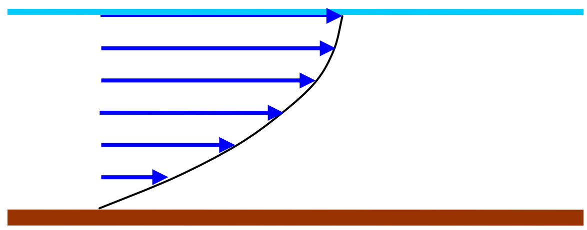
$$\frac{dv(z)}{dz} = -\frac{\rho g\theta}{\eta} z + C ,$$

and $\frac{d}{dz} v(z = h) = 0$ gives $C = \frac{\rho g\theta h}{\eta}$.

Putting this value for C *into* v(z) *we have the final result:*

$$v(z) = \frac{\rho g\theta}{2\eta} z(2h - z).$$

This velocity profile $v(z)$ is half the top part of a parabola:



Total River Flow

Knowing the velocity profile $v(z)$ enables us to compute the *total flow* of water in the river. As explained earlier, we're assuming a wide river having uniform depth, ignoring the slowdown near the edges of the river, taking the same $v(z)$ all the way across. We'll calculate the flow across *one meter of width* of the river, so the total flow is our result multiplied by the river's width.

The flow contribution from a single layer of thickness Δz at height z is $v(z)\Delta z$ cubic meters per second across one meter of width. The total flow is the sum over all layers. In the limit of many infinitely thin layers, that is, $\Delta z \to 0$, the sum becomes an integral, and the total flow rate

$$I = \int_0^h v(z)dz = \int_0^h (\rho g\theta / 2\eta)z(2h - z)dz = (\rho g\theta / 3\eta)h^3$$

in cubic meters per second per meter of width of the river.

It is worth thinking about what this result means physically. The interesting part is that the flow is proportional to h^3, *where* h *is the depth of the river. So, if there's a storm and the river is twice as deep as normal, and flowing steadily, the flow rate will be* eight times *normal.*

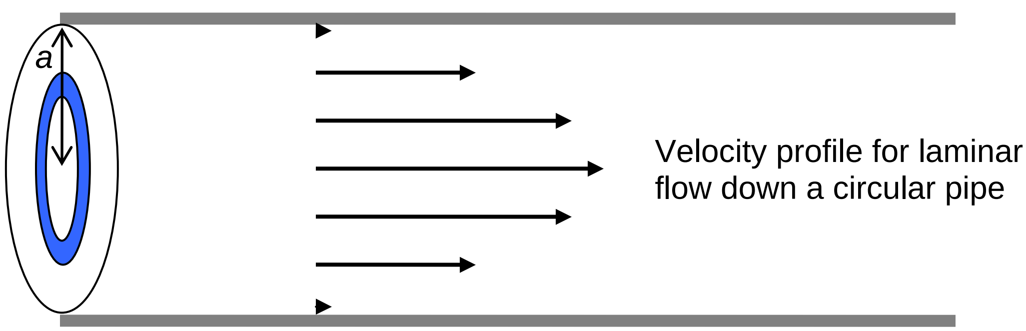
Exercise*: plot on a graph the velocity profiles for two rivers, one of depth* h *and one* 2h, *having the same values of* ρ, g, θ and η. *What is the ratio of the* surface *velocities of the two rivers? Suppose that one meter below the surface of one of the rivers, the water is flowing 0.5 m.sec*$^{-1}$ *slower than it is flowing at the surface. Would that also be true of the other river?*

Flow down a Circular Tube (Poiseuille Flow)

The flow rate for smooth flow through a pipe of circular cross-section can be found by essentially the same method. (This was the flow pattern analyzed by Poiseuille and used by him to confirm Newton's postulate of fluid flow behavior being governed by a coefficient of viscosity.)

In the pipe, the flow is fastest in the middle, and the water in contact with the pipe wall (like that at the river bed) doesn't flow at all. The river's flow pattern was most naturally analyzed by thinking of flat layers of water, all the water in one layer having the same speed. What would be the corresponding picture for flow down a pipe? Here all the fluid *at the same distance from the center* moves down the pipe at the same speed—instead of flat layers of fluid, we have concentric hollow cylinders of fluid, one inside the next, with a tiny rod of the fastest fluid right at the center. This is again laminar flow, even though this time the "sheets" are rolled into tubes. The blue circular area on the cross-section of the pipe shown below represents one of these cylinders of fluid—all the fluid between r and $r+\Delta r$ from the central line.

Each of these hollow cylinders of water is pushed along the pipe by the pressure difference between the ends of the pipe. Each feels viscous forces from its two neighboring cylinders: the next bigger one, which surrounds it, tending to slow it down, but the next smaller one (inside it) tending to speed it up. Writing down the differential equation is a little more tricky that for the river, because we must take into account that the two surfaces of the hollow cylinder (inside and outside) have different areas, $2\pi rL$ and $2\pi\left(r+\Delta r\right)L$. It turns out that the velocity profile is again parabolic: the details are given below.



Circular Pipe Flow: Mathematical Details

Suppose the pipe has radius a, length L and pressure drop ΔP,

$$\text{pressure drop per meter } = \Delta P / L.$$

Let us focus on the fluid in the cylinder between r and $r+\Delta r$ from the line down the middle, and we'll take the cylinder to have *unit length*, for convenience.

The pressure force maintaining the fluid motion is the difference between pressure x area for the two ends of this one meter long hollow cylinder:

$$\text{net pressure force } = \frac{\Delta P}{L} \cdot 2\pi r \Delta r.$$

(We're assuming $\Delta r \ll r$, since we'll be taking the $\Delta r \to 0$ limit, so the end area $\cong 2\pi r \Delta r$. The equality becomes exact in the limit.)

This force exactly balances the difference between the outer surface viscous drag force from the slower surrounding fluid and the inner viscous force from the central faster-moving fluid, very similar to the situation in the previous analysis of river flow.

Using $F / A = \eta dv(z) / dz$, and remembering that the inner and outer surfaces of the cylinder have slightly different areas, the force equation is:

$$\frac{\Delta P}{L} \cdot 2\pi r \Delta r = 2\pi (r + \Delta r) \eta \frac{dv(r + \Delta r)}{dr} - 2\pi r \eta \frac{dv(r)}{dr}.$$

Rearranging,

$$\begin{aligned} \frac{\Delta P}{L} \cdot r &= \frac{(r + \Delta r) \eta \frac{dv(r + \Delta r)}{dr} - r\eta \frac{dv(r)}{dr}}{\Delta r} \\ &= \eta \frac{d}{dr} \left(r \frac{dv}{dr} \right) \end{aligned}$$

in the limit $\Delta r \to 0$, remembering the definition of the differential (see the similar analysis above for the river).

This can now be integrated to give

$$r \frac{dv}{dr} = \frac{\Delta P}{\eta L} \cdot \frac{r^2}{2} + C$$

where C is a constant of integration. Dividing both sides by r and integrating again

$$v(r) = \frac{\Delta P}{\eta L} \cdot \frac{r^2}{4} + C \ln r + D.$$

The constant C *must be zero, since physically the fluid velocity is finite at* r *= 0. The constant* D *is determined by the requirement that the fluid speed is zero where the fluid is in contact with the tube, at* r = a*.*

The fluid velocity is therefore

$$v(r) = \frac{\Delta P}{\eta L} \cdot \frac{(a^2 - r^2)}{4}.$$

To find the total flow rate *I* down the pipe, we integrate over the flow in each hollow cylinder of water:

$$I = \int_0^a 2\pi r v(r)\, dr = \int_0^a 2\pi \frac{\Delta P}{\eta L} \cdot \frac{\left(a^2 r - r^3\right)}{4} dr = \frac{\pi \Delta P}{8\eta L} \cdot a^4$$

in cubic meters per second.

Notice the flow rate goes as the *fourth power* of the radius, so doubling the radius results in a *sixteen-fold* increase in flow. That is why narrowing of arteries is so serious.

Using Dimensions

M, L and T: all physics equations must have the same dimensions on both sides. This can be exploited to arrive at some interesting predictions without doing much math—for example, that the smooth flow rate through a circular pipe goes as the fourth power of the radius.

Some of the most interesting results of hydrodynamics, such as the sixteen-fold increase in flow down a pipe on doubling the radius, can actually be found without doing any calculations, just from dimensional considerations.

We symbolize the "dimensions" *mass*, *length* and *time* by *M*, *L*, *T*. We then write the dimensions of other physical quantities in terms of these. For example, velocity has dimensions LT^{-1}, and acceleration LT^{-2}.

We shall use *square brackets* [] to denote the dimensions of a quantity, for example, for velocity, we write $[v] = LT^{-1}$. Force must have the same dimensions as mass times acceleration, so $[F] = MLT^{-2}$. This "dimensional" notation does *not* depend on the units we use to measure mass, length and time.

***All equations in physics must have the same dimensions on both sides*.**

We can see from the equation defining the coefficient of viscosity η, $F/A = \eta v_0/d$, (the left hand side is force per unit area, the right hand v_0/d is the velocity gradient) that

$$[\eta] = ([F]/[A]) \cdot [d]/[v] = \left(MLT^{-2}/L^2\right) \cdot L/LT^{-1} = ML^{-1}T^{-1}.$$

How can thinking dimensionally help us find the flow rate I *through a pipe? Well, the flow itself, say in cubic meters per second, has dimensions* $[I] = M^3T^{-1}$. *What can this flow depend on?*

The physics of the problem is that the pressure difference ΔP between the ends of the pipe of length L is doing work overcoming the viscous force. *The* only *parameters determining the flow are therefore: the pressure gradient,* $\Delta P/L$*, the viscosity* η*, and the radius of pipe cross section* a*. Note here that we are assuming the flow is* steady*—no acceleration—so the mass, or more precisely density, of the fluid plays no part. Of course, if the flow is downward, the density has an* indirect *role in that the weight of the fluid generates the pressure gradient, but we've already included the pressure as a parameter.*

Therefore,

$$\text{Flow } I = f\left(\Delta P/L, \eta, a\right)$$

where f is some function we don't know, but we do *know that the two sides of this equation must match dimensionally, so* f *must have the same dimensions as* I*, that is,* L^3T^{-1}.

Now ΔP*, a pressure, has dimensions* $\left[F\right]/\left[A\right] = MLT^{-2}L^{-2}$ *so* $\Delta P/L$ *has dimensions* $ML^{-2}T^{-2}$.

The other variables in f *have dimensions* $\left[\eta\right] = ML^{-1}T^{-1}$ *(from above) and [*a*]* = L.

The game is to put these three variables (or powers of them) together to give a function f *having the dimensions of flow, that is,* L^3T^{-1}*, otherwise the above equation must be invalid.*

The first thing to notice is that there is no M *term in flow, and none in* a *either, so* $\Delta P/L$ *and* η *must appear in the equation in such a way that their* M *terms cancel, that is, one divides the other.*

We know of course that increased pressure increases the flow, so they must appear in the combination $\Delta P/L\eta$*. This gets rid of* M*. The next task is to put this combination, which has itself dimensions* $MLT^{-2}L^{-2}/ML^{-1}T^{-1} = L^{-1}T^{-1}$*, together with [*a*]* = L*, to get a quantity with the dimensions of flow,* L^3T^{-1}*. The unique choice is to multiply* $\Delta P/L\eta$ *by* a^4.

We therefore conclude that the flow rate through a circular pipe must be given by:

$$I = C\left(\Delta P/L\eta\right)a^4.$$

This is certainly much easier than solving the differential equation and integrating to find the flow rate! The catch is the unknown constant C *in the equation—we can't find that without doing the hard work. However, we* have *established from this dimensional argument that the flow rate increases by a factor of 16 when the radius is doubled.*

It should be noted that this conclusion does *depend on the validity of the assumptions made—in particular, that the flow is uniform and in straight streamlines. At sufficiently high pressure, the flow becomes turbulent. When this happens, the pressure causes the fluid to bounce around inside the pipe, and the flow pattern will then depend also on the* density *of the fluid, which was irrelevant for the slow laminar flow, and the reasoning above will be invalid.*

Exercise*: derive the depth dependence of the steady flow of a wide river under gravity. (Note: The appropriate flow rate is cubic meters per second* per meter of width of the river.*)*

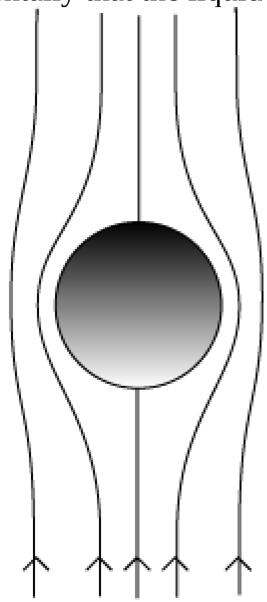
So dimensional analysis cannot give overall dimensionless constants, but can predict how flow will change when a physical parameter, such as the pressure or the size of the pipe, is altered. We've shown above how it rather easily gives a nonobvious result, the a^4 *dependence of flow on radius, which we found earlier with a good deal of work. But as we shall see, dimensional analysis can also illuminate the essential physics of flow problems where exact mathematical analysis is far more difficult, such as Stokes' Law in the next lecture, and help us understand how the nature of fluid flow changes at high speeds.*

Dropping the Ball (Slowly)

Dropping a small ball through a very viscous fluid: a dimensional prediction of the dependence of speed on radius, and an experiment with glycerin.

Stokes' Law

We've seen how viscosity acts as a frictional brake on the rate at which water flows through a pipe, let us now examine its frictional effect on an object falling through a viscous medium. To make it simple, we take a sphere. If we use a very viscous liquid, such as glycerin, and a small sphere, for example a ball bearing of radius a millimeter or so, it turns out experimentally that the liquid flows smoothly around the ball as it falls,

with a flow pattern like: 

(The arrows show the fluid flow as seen by the ball. This smooth flow *only takes place for fairly slow motion*, as we shall see.)

If we knew mathematically precisely how the velocity in this flow pattern varied near the ball, we could find the total viscous force on the ball by finding the velocity gradient near each little area of the ball's surface, and doing an integral. But actually this is quite difficult. It was done in the 1840's by Sir George Gabriel Stokes. He found what has become known as Stokes' Law: the drag force F on a sphere of radius a moving through a fluid of viscosity η at speed v is given by:

$$F = 6\pi a\eta v.$$

Note that this drag force is *directly proportional to the radius*. That's not obvious—one might have thought it would be proportional to the cross-section area, which would go as the square of the radius. The drag force is also *directly proportional to the speed*, not, for example to v^2.

Understanding Stokes' Law with Dimensional Analysis

Is there some way we could see the drag force must be proportional to the radius, and to the speed, without wading through all of Sir George's mathematics? The answer is yes—by using **dimensions**.

First we must ask: what can this drag force depend on?

Obviously, it depends on the *size* of the ball: let's say the radius is a, having dimension L.

It must depend on the *speed* v, which has dimension LT^{-1}.

Finally, it depends on the *coefficient of viscosity* η which has dimensions $ML^{-1}T^{-1}$.

The drag force F has dimensions $[F] = MLT^{-2}$: what combination of $[a] = L$, $[v] = LT^{-1}$ and $[\eta] = ML^{-1}T^{-1}$ will give $[F] = MLT^{-2}$?

It's easy to see immediately that F must depend linearly on η, that's the only way to balance the M term.

Now let's look at F/η, which can only depend on a and v. $[F/\eta] = L^2T^{-1}$. The only possible way to get a function of a, v having dimension L^2T^{-1} is to take the product av.

So, the dimensional analysis establishes that the drag force is given by:

$$F = Ca\eta v$$

where C is a constant that cannot be determined by dimensional considerations.

Experimental Check

We can check this result by dropping small steel balls through glycerin. We choose glycerin because it has a very high viscosity, so the balls fall slowly enough for us to be able to time them.

One problem is that the viscosity of glycerin is *very* temperature dependent, being 1.49 Pa.sec at 20°C, and 0.95 Pa.sec at 25°C. We measured the temperature of our glycerin to be 23°C, so we assumed its viscosity was 1.17 Pa.sec., just taking a linear interpolation. We used a ball of radius 1.2mm, weighing 0.05 grams. On dropping it through the glycerin, and allowing some distance for it to reach a steady speed, we found it fell 25cm. in 11.1 seconds, a speed of 0.022 m sec^{-1}. (I got these numbers in a trial run preparing for class.) So the drag force should be:

$$F = 6\pi a\eta v \cong 18.8 \times 1.2 \times 10^{-3} \times 1.17 \times 2.2 \times 10^{-2} \cong 6 \times 10^{-4}\,\text{N}.$$

If the ball is dropping at a steady speed, this force should just balance the weight of the ball. The mass is 0.05grams, which is 5.10^{-4} N. But we should also have subtracted off a buoyancy force, which would get this closer to 4.10^{-4} N. Since we think our measurements of radius, mass and speed were fairly accurate, the viscosity was evidently less than we thought. Bearing in mind that it drops by 10% for each one-degree rise in temperature, most likely it was not at a uniform temperature, or our measurement of temperature was inaccurate.

We checked the *dimensional* prediction by dropping a ball of exactly *twice* the radius. It fell in exactly *one-quarter* of the time.

This confirms the correctness of the dimensional analysis, because once the ball has reached terminal velocity v_{term}, and therefore is no longer accelerating, it must feel zero net force. At this stage, the forces of viscous drag and weight must be in balance:

$$Ca\eta v_{\text{term}} = (4/3)\pi a^3 \rho g.$$

It follows that for two balls of the same density ρ, after canceling a from each side, the ratio of their terminal velocities is the *square* of the ratio of their radii, a ball with radius $2a$ will fall four times faster than a ball with radius a. This is what we found experimentally.

Exercise: Assuming the flow pattern in the diagram above has the same proportions for different radii (so for a larger radius ball it's the same pattern magnified), how does the fluid *velocity gradient* near the "equator" of the ball change on going from a ball of radius a to one of radius $2a$? (Assume the two balls are falling through the fluid at the same speed.) Argue that most of the viscous drag on the sphere takes place in a band surrounding the equator (so, a band shaped like the tropical zone on the earth). From this, make plausible that the total viscous drag will be proportional to the sphere's radius, not to the square of the radius.

Stokes' Law and the Coffee Filters

Another experiment, this time dropping coffee filters through air, with a very different result—but also predicted dimensionally! The Reynolds number: the dimensionless ratio of inertial drag to viscous drag.

A Problem

We found that Stokes' Law, which we derived in the form

$$F_{\text{drag}} = Ca\eta v$$

from purely dimensional considerations (Stokes did the hard part of proving that $C = 6\pi$) correctly predicted that for two small steel balls, one having a radius exactly twice the other, the bigger one would fall through a fluid four times faster (it had eight times the weight, and twice the drag force for the same velocity, and the drag force is proportional to the velocity).

Now let us ask what Stokes' Law predicts for the following *coffee filter experiment*:

If we drop a single coffee filter, it reaches a terminal velocity of about 0.8 meters per sec after falling less than a meter. If we drop a stack of four close packed filters, the terminal velocity clocks in at about 1.6 meters per sec.

That is to say, the stack of *four* filters has a terminal velocity *twice* that of a single filter. Now at terminal velocity the drag force is exactly balancing the weight of the object falling. The stack of four filters is almost indistinguishable in shape and size from the single filter, so it's difficult to believe there's any significant difference in the air flow pattern round the falling filters for the same speed. Therefore the Stokes' drag from the air friction should be the same $Ca\eta v$ for both. (We can't say $C = 6\pi$, that was derived for a falling *sphere*, but the *dimensional* argument should still be working.) Yet this implies that the terminal velocity of the stack of four filters should be *four times* the terminal velocity of the single filter!

What is wrong with our dimensional analysis? It worked brilliantly for the little steel balls, but seems to have flunked the coffee filter test. In what respect are these two experiments different?

Another Kind of Drag Force

Perhaps the best way to see what is wrong is to do the steel ball experiment on a completely different scale. Let us imagine dropping a cannonball from an airplane. This will also reach a terminal velocity, but at hundreds of miles an hour. However, in contrast to the steel balls in glycerin experiment, it turns out that this time the viscous drag is not the important effect. *At high speeds, most of the work done by the falling body is in just pushing the air out of the way*.

Let us estimate how much force the cannonball exerts on the air pushing it out of its path. Suppose the cannonball is falling at steady speed v, and it has radius a. Then it has to move aside a volume $\pi a^2 v$ of air per second, and this air will be moved at a speed of order of magnitude v. Therefore, the rate at which the cannonball imparts momentum to the air (which was previously at rest) is of order $\pi \rho a^2 v^2$ per second. But the rate of change of momentum per second is just the force, so the cannonball is pushing the air with a force of order $\pi\rho a^2 v^2$. By Newton's Third Law, Action = Reaction, this is also the drag force the cannonball experiences as it falls at v.

Exercise: Assuming the drag force depends only on v, a, and the density of air ρ, use a dimensional argument to show it must have this form.

So What is the *Real* Drag Force?

Using purely dimensional considerations, we have derived two quite different formulas for the drag force on a sphere falling through a fluid:

Viscous drag force:

$$F_{\text{viscous}} = Ca\eta v$$

and inertial drag force:

$$F_{\text{inertial}} = C'\rho a^2 v^2.$$

We call the second "inertial" because it arises from just pushing the still air out of the way, and would be the same if the air had no viscosity at all.

The truth is that the two different derivations we have presented above for these two different drag forces are both too simple. In fact, in real situations, *both* types of forces are present. This does not mean, though, that we can simply add the forces with suitable coefficients—the general situation is far more complicated. However, it can be described mathematically by a complicated differential equation, the Navier-Stokes equation. The good news is that the solutions to this equation for a given flow configuration, such as flow past a sphere, or flow past a wing, can be classified in terms of a single dimensionless parameter, the Reynolds number.

The Reynolds number is just the ratio of the inertial drag to the viscous drag:

$$N_{\text{R}} = 2a\rho v/\eta.$$

The factor of 2 is the standard definition of the Reynolds number—this is just a matter of convention, it is of course not fixed by the dimensional arguments. And the Reynolds number is **dimensionless**: it's the ratio of two forces, so will be the same in any system of units!

The theoretical prediction from the Navier-Stokes equation that the flow pattern in a given geometry depends only on the Reynolds number is well established experimentally, and makes it possible to find how air flows around an airplane in flight by testing a scale model in a wind tunnel, adjusting wind speed to get the same Reynolds number.

Stokes' Law for a falling sphere is found experimentally to be reasonably accurate for N_{R} less than or of order 1.

Reference: The derivation of Stokes' Law (the 6π) can be found, for example, in G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge, 1967, 2000.

Fluids Fact Sheet

Pressure

1 Pascal = 1 Newton/m^2.
1 atm. = 101.3 kPa = 1.013 bar = 760 mm Hg (760 Torr) = 14.7 lb/in^2.

Some Densities

Material	**Density in kg/m^3**
Air at 760 mm, 0°C (STP)	1.29
Air at 760 mm, 100°C	0.944
Hydrogen (STP)	0.090
Helium (STP)	0.179
Water	1000
Sea Water	1025
Olive Oil	800 - 920
Mercury	13600
Aluminum	2700
Iron	7860
Silver	10500
Gold	19300

Some Viscosities: Liquids

Fluid	**Viscosity in mPa.s**
Water at 0°C	1.79
Water at 20°C	1.005
Water at 100°C	0.28
Glycerin at 0°C	12070
Glycerin at 20°C	1410
Glycerin at 30°C	612
Glycerin at 100°C	14.8
Mercury at 20°C	1.55
Mercury at 100°C	1.27
Motor Oil SAE 30	200
Motor Oil SAE 60	1000
Ketchup	50,000

Some Viscosities: Gases

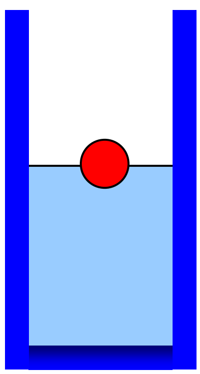
Gas	Viscosity in 10^{-6} Pa.s
Air at 0°C	17.09
Air at 20°C	18.08
Air at 100°C	21.30
Hydrogen at 0°C	8.4
Helium at 0°C	18.6
Xenon at 0°C	21.2

Physics 152: Homework Problems on Fluids

1. Atmospheric pressure varies from day to day, but **1 atm** is defined as 1.01×10^5 Pa. Calculate how far upwards such a pressure would force a column of water in a "water barometer". (That is, a long inverted glass tube, the top end sealed and having vacuum inside above the water surface, the bottom end immersed under water in a bowl of water, the other water surface in the bowl being open to the atmosphere.)

2. The density of air at room temperature is about 1.29 kg/m^3. Use this together with the definition of 1 atm as 1.01×10^5 Pa. to find the constant C in the Law of Atmospheres $P(h) = P_0 e^{-Cgh}$. Use your result to estimate the atmospheric pressure on top of the Blue Ridge (say 4000 feet), Snowmass (11,000 feet) and Mount Everest (29,000 feet).

3. As a practical matter, how would you measure the density of air in a room? Actually, Galileo did this in the early 1600's. Can you figure out how he managed to do it? (His result was off by a factor of two, but that was still pretty good!)



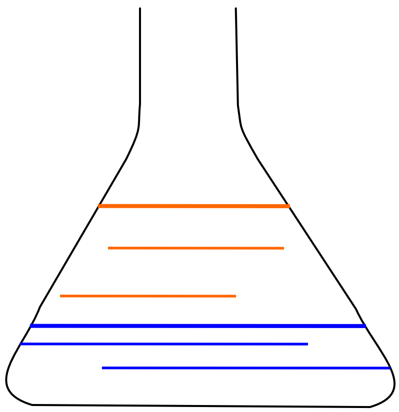
4. A ball is floating in water, exactly half submerged, as shown in the diagram. Oil is now poured gently onto the water, so that it does not mix, but forms a layer above the water. The oil completely covers the ball. Is the center of the ball now above, below or at the water-oil boundary?

5. A weather balloon is made of latex, filled with helium (but not stretched) so that it can just lift an instrument package of 10 kg., including the weight of the balloon.

(a) What is the volume of the balloon?

(b) The balloon lifts the package to a height of 30 km. Assume the balloon has stretched to a larger size, but is still intact. Assume also (falsely!) that the air temperature is the same. How big is the balloon at 30 km.?

(c) How does the lifting capacity of the balloon at 30 km compare with that at ground level?



6. I make salad dressing in a **conical** bottle, wide at the base, steadily narrowing up to the neck. I pour in 100 cc of vinegar, then add 100 cc of olive oil. The oil is a layer on top of the vinegar. Then I shake it vigorously, so the two liquids are completely mixed together.

How (if at all) does the pressure on the bottom after shaking differ from the pressure on the bottom before shaking?

7. If an ice cube is floating in a glass filled to the brim with water, what happens to the water level as the ice cube melts? Would your answer be the same if the ice cube were stuck to the glass and fully submerged?

8. (a) State the Law of Atmospheres: the equation relating the density of air to height above the Earth's surface. (You don't have to derive it, just state it.)

$$P = P_0 e^{-Cgh}$$

(b) Given that the density of the atmosphere at 3,500 meters is approximately 2/3 that at sea level, sketch a rough graph of density with height up to 14,000 meters. (Take sea level density 1.3 kg/m^3.)

(c) In 1804, a French chemist, Gay-Lussac, went up in a balloon to about 7,000 meters to check the chemistry of the atmosphere. The balloon was filled with hydrogen. Make a very rough estimate of the size of the balloon. (Guess Gay-Lussac's weight, assume the balloon weighed three times as much as he did.)

9. A modern hot air balloon can weigh about 4 tons, fully loaded. The maximum operating temperature for the enclosed hot air is 120 degrees C. (Otherwise the nylon interior deteriorates rapidly.) Assume the enclosed hot air is at 100 degrees, and the outside atmosphere is at zero degrees. If the balloon can just lift off, how big is it?

10. You have a solid metal cylinder, of height h, cross section area A, and density ρ, standing on its end on a table. What is the pressure on the table underneath the cylinder? Suppose you now heat the cylinder, and it expands by the same percentage in all directions (so the shape and proportions are unchanged), the height increasing to $h+\Delta h$. What is the pressure on the table now?

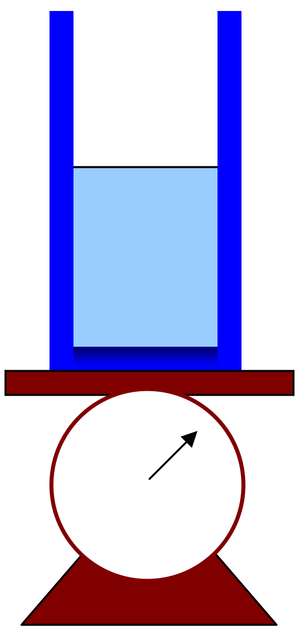
11. Assume you are walking around underwater, breathing through a flexible plastic tube, the other end held on the surface by some floating object. The object is to estimate how far underwater it's safe for you to go.

(a) Assume first you are out of the water, lying flat on your back, with a weight evenly distributed across your chest. What weight would cause you difficulty in breathing?

(b) Now, make a guess as to the area of your chest over the lungs, and deduce at what depth the water pressure on your chest would become dangerous.

12. Figure out what total area of one tire of your automobile is in contact with the road when the vehicle is parked on level ground.

13. What is the approximate pressure your shoes exert on the ground when you're standing still? Give a ballpark estimate for the pressure exerted by a woman balancing on one high heel.



14. A beaker containing water is placed on a spring scale as shown. Next, a cork of mass 10 grams, and density 200 kg/m^3, is gently floated on the water, not touching the sides of the container.

(a) How does that change the scale reading?

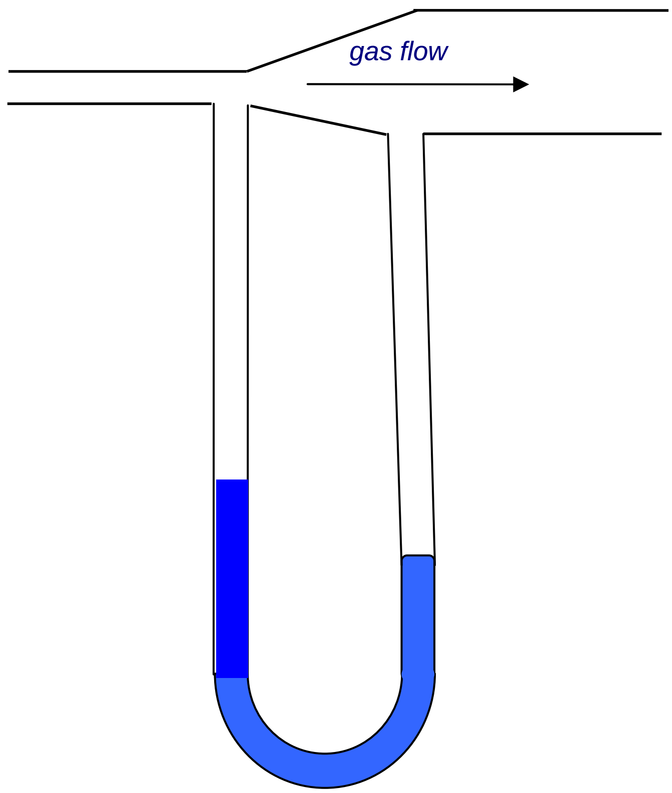
(b) A thin rod, of negligible volume, is now used to push the cork underwater, again without it touching the container? Does that change the scale reading? Explain your reasoning.

(c) What if the cork is held underwater by a thin string attached to a small hook in the middle of the base of the container? What does the scale read in that case? Does the tension in the string play a role?

15. In a car going down a highway at a steady velocity, a child has a helium-filled balloon on a string, the balloon is at rest directly above the child, not touching the roof of the car. Now the driver accelerates. How does the balloon move? Explain your reasoning.

16. (a) Explain how a *nonvertical* jet can keep a beach ball in the air.

(b) Write down your best guesses for the weight and size of the (very light!) beach ball, etc., then use them to make a *ballpark* estimate of the speed of the air in the jet holding the ball up.



*A Venturi flow **meter** is a manometer with the two arms connected to places in the flow tube having different cross sections. The fluid flow rate can be figured from the pressure difference registered.*

17. In the flow meter shown, air flows from a pipe of cross sectional area 10 sq cm into one of cross sectional area 40 sq cm. The manometer contains water, the height difference between the two arms is 5 cm. What is the rate of air flow?

18. A large beer keg of height H and cross-section area A_1 is filled with beer. The top is open to atmospheric pressure. At the bottom is a spigot opening of area A_2, which is much smaller than A_1.

(a) Show that when the height of the beer is h, the speed of the beer leaving the spigot is approximately $\sqrt{(2gh)}$.

(b) Show that for the approximation $A_2 \ll A_1$, the rate of change of the height h of the beer is given by

$$\frac{dh}{dt} = -\frac{A_2}{A_1}(2gh)^{1/2}$$

(c) Find h as a function of time if $h = H$ at $t = 0$.

(d) Find the total time needed to drain the keg if $H = 2$ m, $A_1 = 0.8$ m^2, and $A_2 = (10^{-4})A_1$.

19. (a) Write down the equation that defines the coefficient of viscosity η.

(b) By balancing dimensions on both sides of your equation, find the dimensions $M^?L^?T^?$ of η.

(c) What are the dimensions of flow rate of blood through an artery?

(d) What are the dimensions of the pressure gradient (pressure drop per unit length) in an artery?

(e) Assuming that the blood flow rate depends only on the pressure gradient, the viscosity and the radius R of the artery, give a proof using dimensions, and showing all your steps, that the flow rate is proportional to R^4.

(f) If plaque build up in an artery reduces its radius by 10%, estimate how much the blood flow is reduced if the blood pressure stays the same.

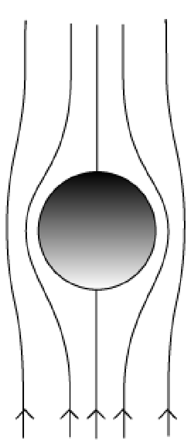
20. A simple hot tub has vertical sides, water depth one meter and area 4 square meters. It has a drain at the bottom, a circular opening of radius 3 cm. When the plug is pulled, it drains gravitationally (no pump).

(a) What is the speed of water flow through the circular opening right after the plug is pulled? (Ignore viscosity: assume the water is flowing at this same speed over the whole area of the opening.)
(b) At what speed is the water level in the tub dropping right after the plug is pulled?
(c) How quickly is the level in the tub dropping when the tub is half empty?
(d) How long does it take for the tub to empty completely?

21. *Suppose a mass* m *of fluid moving at* v_1 *in the* x*-direction mixes with a mass* m *moving at* v_2 *in the* x*-direction. Momentum conservation tells us that the mixed mass 2*m *moves at* $\frac{1}{2}(v_1 + v_2)$*. Prove that the total kinetic energy has decreased if* v_1, v_2 *are unequal.*

22. Using only dimensional arguments, derive the depth dependence of the steady flow of a wide river under gravity. (Note: The appropriate flow rate is cubic meters per second *per meter of width of the river*.)

23. Plot on a graph the velocity profiles for two rivers, one of depth h and one $2h$, having the same values of ρ, g, θ and η. What is the ratio of the *surface* velocities of the two rivers? Suppose that one meter below the surface of one of the rivers, the water is flowing 0.5 $m.sec^{-1}$ slower than it is flowing at the surface. Would that also be true of the other river?



24. Assuming the flow pattern in the diagram has the same proportions for different radii (so for a larger radius ball it's the same pattern magnified), how does the fluid *velocity gradient* near the "equator" of the ball change on going from a ball of radius a to one of radius $2a$? (Assume the two balls are falling through the fluid at the same speed.)

Argue that most of the viscous drag on the sphere takes place in a band surrounding the equator (so, a band shaped like the tropical zone on the earth). From this, make plausible that the total viscous drag will be proportional to the sphere's radius, *not* to the square of the radius.

25. An aged professor vigorously cleaning a blackboard raises a cloud of chalk dust that takes several minutes to settle. From this information, make some estimate of the size of chalk particles in the cloud. Also state whether the drag force from the air on the particles is mainly viscous or mainly inertial (for air $\eta = 2\times10^{-5}$ Pa.sec.).

26. Imagine a foggy day when the air is still. Actually, the fog is made up of tiny spherical droplets of water. The fog obviously doesn't fall to the ground very fast. Use this fact to make some estimate of the probable maximum size of the droplets.

27. (a) A room freshener sprays out a mist of tiny water droplets (containing some odor neutralizer which has a negligible effect on the density of the droplets). It is claimed that the mist will stay in the air for 30 minutes. Assuming the air in the room is still, and neglecting possible evaporation from the droplets, figure out from this the approximate size of the droplets.

(b) Justify any formula you use by finding a Reynolds number.

28. After a storm, some rivers and lakes become muddy. Assume the mud particles have the density of ordinary rock, say 2500 kg per cubic meter, and assume they are spherical to a good approximation. If the still water in a lake one meter deep takes two days for the mud to settle to the bottom, use Stokes' Law to give a ballpark estimate of the size of the mud particles.

29. By staring at a fizzy drink, make some estimate of the size and speed of a typical bubble rising to the top. Pick the smallest one you can comfortably see and try to time its speed—or make an estimate. What, very approximately, is the Reynolds number for the flow of liquid around the bubble? Is the bubble impeded in its rise mainly by viscous or by inertial forces? (Use the value of η for water at room temperature, η =1mPa.sec.)

30. Estimate what would be the maximum height of a wall you could jump off and be pretty certain of landing without breaking anything (I'm not responsible if you try this!), and, from that estimate, figure out a reasonable size for the diameter of a parachute.

31. Take the following simple model of a skydiver: suppose that as he falls, all the air is his direct path is deflected sideways just as if it bounced off him, imagining his downward profile to be V-shaped. Ignore viscosity. By making reasonable estimates of the size and weight of the skydiver, figure out his terminal velocity.

32. (a) Write down the definition of the Reynold's number.

(b) Give a *ballpark* estimate for the Reynold's number for airflow around a car moving down the highway at 20 $m \cdot sec^{-1}$ on a day when there is no wind.

(c) From your estimate above, how significant is the viscosity of the air in the drag force?

(d) If we assume the drag force F only depends on the density of the air ρ, the dimensions of the car a, and the speed of the car v, show from dimensional arguments that $F = C\rho a^2 v^2$, where C is a constant.

(e) Take $C = 0.2$ and give an estimate of this force for your car. You can take a value for ρ accurate to *one* significant figure. State clearly what you take for a^2.

(f) What power in kilowatts is the car expending in overcoming this drag force at 20 $m \cdot sec^{-1}$?

(g) How much would this power go up if the speed were doubled?

33. (a) Prove by a dimensional argument that the drag force on a falling sphere caused by pushing air aside has the form $F_{\text{drag}} = C\rho a^2 v^2$, where C is a constant, ρ is the air density, a is the *radius* of the sphere, and v its speed.

(b) You are jogging a 7½ minute mile in light rain. There is no wind, the rain appears to you to be coming down towards you at a 45° angle. How fast is the rain falling, in mph and in m sec^{-1}? (1 mile = 1.6 km.)

(c) For raindrops, the C in the formula in part (a) is approximately 1. Use this to find the size of the raindrops in part (b). (Take air density = 1.3 kg/m^3, g = 10 $m.sec^{-2}$.)

(d) Find the Reynolds number. Was it OK to neglect viscous drag?

(e) Would you expect a different answer for raindrop size if you were jogging under identical conditions, and seeing the same angle of fall, but at a high altitude resort? Why?