Lectures on Gravity

Michael Fowler, University of Virginia, Physics 152 Notes, May, 2007

Discovering Gravity

Terrestrial Gravity: Galileo Analyzes a Cannonball Trajectory

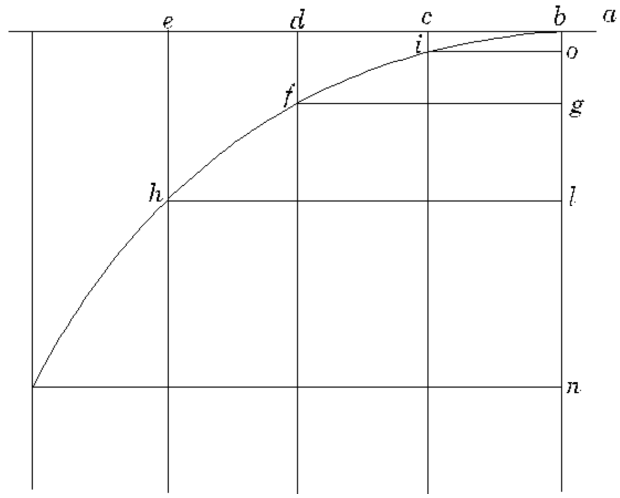
From the earliest times, gravity meant the tendency of most bodies to fall to earth. In contrast, things that leaped upwards, like flames of fire, were said to have “levity”. Aristotle was the first writer to attempt a quantitative description of falling motion: he wrote that an object fell at a constant speed, attained shortly after being released, and heavier things fell faster in proportion to their mass. Of course this is nonsense, but in his defense, falling motion is pretty fast—it’s hard to see the speed variation when you drop something to the ground. Aristotle most likely observed the slower motion of things falling through water, where buoyancy and fluid resistance dominate, and assumed that to be a slowed-down version of falling through air—which it isn’t.

Galileo was the first to get it right. (True, others had improved on Aristotle, but Galileo was the first to get the big picture.) He realized that a falling body *picked up speed at a constant rate*—in other words, it had constant *acceleration* (as he termed it, the word means “addition of speed” in Italian). He also made the crucial observation that, if air resistance and buoyancy can be neglected, *all bodies fall with the same acceleration*, bodies of different weights dropped together reach the ground at the same time. This was a revolutionary idea—as was his assertion that it should be checked by *experiment* rather than by the traditional method of trying to decipher what ancient authorities might have meant.

Galileo also noted that if a ball rolls without interference on a smooth horizontal surface, and friction and air resistance can be neglected, it will move with constant speed in a fixed direction—in modern language, its velocity remains constant.

He considered the motion of an object when not subject to interference as its “*natural*” motion. Using his terminology, then, *natural horizontal motion* is motion at *constant velocity*, and *natural vertical motion* is falling at *constant acceleration*.

But he didn’t stop there—he took an important further step, which made him the first in history to derive useful quantitative results about motion, useful that is to his boss, a duke with military interests. The crucial step was the realization that for a cannonball in flight, *the horizontal and vertical motions can be analyzed independently*. Here’s his picture of the path of a horizontally fired cannonball:



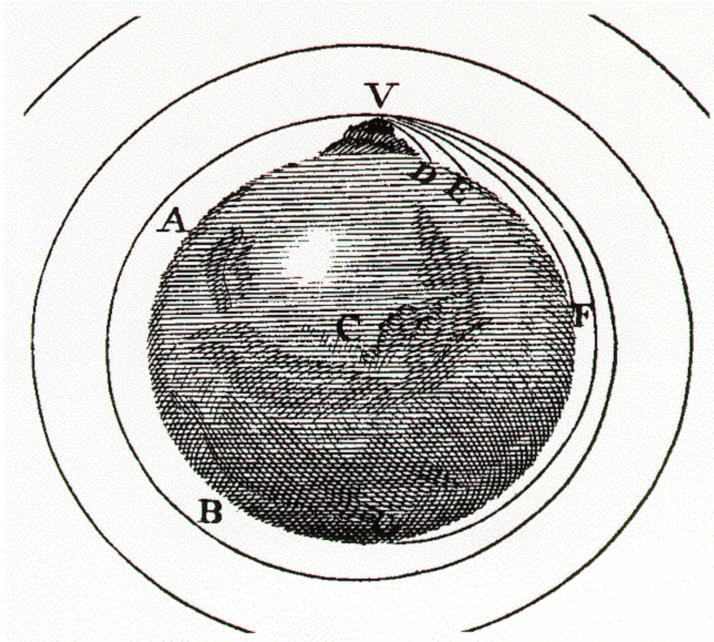
The vertical drop of the cannonball at the end of successive seconds, the lengths of the vertical lines *ci*, *df*, *eh* are the same vertical distances fallen by something dropped from rest. If you drop a cannonball over a cliff it will fall 5 meters in the first second, if you fire it exactly horizontally at 100 meters per second, it will *still* fall 5 meters below a horizontal line in the first second. Meanwhile, its horizontal motion will be at a steady speed (again neglecting air resistance), it will go 100 meters in the first second, another 100 meters in the next second, and so on. Vertically, it falls 5 meters in the first second, 20 meters total in two seconds, then 45 and so on.

Galileo drew the graph above of the cannonball's position as a function of time, and proved the curve was parabolic. He went on to work out the range for given muzzle velocity and *any* angle of firing, much to the gratification of his employer.

Moving Up: Newton Puts the Cannon on a Very High Mountain

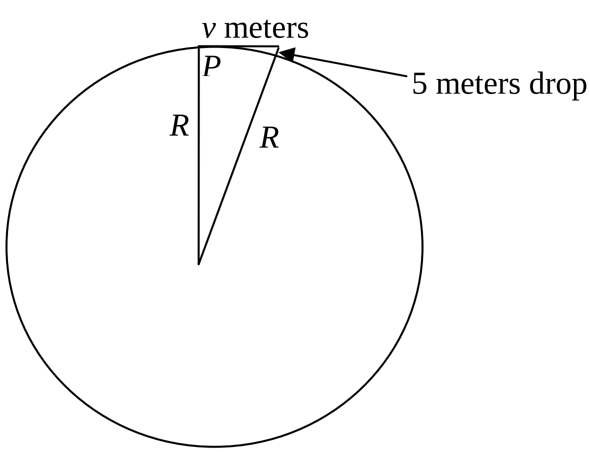
Newton asked the question: what if we put the cannon on a really high (imaginary, of course!) mountain above the atmosphere and fired the cannon *really fast*? The cannonball would still fall 5 meters in the first second (ignoring the minor point that g goes down a bit on a really high mountain), *but* if it's going fast enough, don't forget the curvature of the earth! The surface of the earth curves away below a horizontal line, so if we choose the right speed, after one second the cannonball will have reached a point where the earth's surface itself has dropped away by 5 meters below the originally horizontal straight line. In that case, the cannonball won't have lost any height at all—defining "height" as distance above the earth's surface.

Furthermore, "vertically down" has turned around a bit (it means perpendicular to the earth's surface) so the cannonball is still moving "horizontally", meaning moving parallel to the earth's surface directly beneath it. And, since it's above the earth's atmosphere, it won't have lost any speed, so exactly the same thing happens in the next second, and the next—it therefore goes in a circular path. Newton had foreseen how a satellite would move—here's his own drawing, with VD, VE and VF representing the paths of successively faster shots:



Newton's brilliant insight—the above picture—is fully animated in my applet here.
We can find how fast the cannonball must move to maintain the circular orbit by using Pythagoras' theorem in the diagram below (which grossly exaggerates the speed so that you can see how to do the proof).

The cannonball fired from point *P* goes *v* meters horizontally in one second and drops 5 meters vertically, and, if *v* has the right value, the cannonball will still be the same distance *R* from the earth's center it was at the beginning of the second. (Bear in mind that *v* is actually about a thousandth of *R*, so the change in the direction of "down" will be imperceptible, not like the exaggerated figure here.)



Knowing that the radius of the earth R is 6400 km, there is enough information in the above diagram to fix the value of v. Notice that there is a right angled triangle with sides R and v and hypoteneuse $R+5$. Applying Pythagoras' theorem,

$$(R+5)^2 = R^2 + v^2,$$

$$R^2 + 10R + 25 = R^2 + v^2.$$

Newton knew (in different units) that R = 6400 km, so the 25 in the above equation can be neglected to give:

$$v^2 = 10R = 10 \times 6400 \times 1000, \text{ so } v = 8000.$$

The units for v are of course meters per second, on our diagram we show v as a distance, that traveled in the first second.

So the cannonball must move at 8 km per second, or 5 miles per second if its falling is to match the earth's curvature—this is 18,000 mph, once round the earth in a little less than an hour and a half. This is in fact about right for a satellite in low earth orbit.

Onward into Space: The Cannonball and the Moon

It occurred to Newton one day (possibly because of a falling apple) that this familiar gravitational force we experience all the time here near the surface of the earth might extend outwards as far as the moon, and in fact might be the reason the moon is in a circular orbit. The radius of the moon's orbit (384,000 km) and its speed in orbit (about 1 km per second) had long been known (see my notes here if you're interested in how it was measured), so it was easy to find, using the same Pythagorean arguments as used for the cannonball above, that the moon "falls" 1.37 millimeters below a straight line trajectory in one second.

That is to say, the ratio of the moon's acceleration towards the center of the earth to the cannonball's is 1.37/5000, or about 1/3600.

But the radius of the moon's orbit is about *60 times greater than the cannonball's* (which is just the radius of the earth, approximately). Since $60 \times 60 = 3600$, Newton concluded that the gravitational force decreased with distance as $1/r^2$.

Newton's Universal Law of Gravitation

Newton then boldly extrapolated from the earth, the apple and the moon to everything, asserting his Universal Law of Gravitation:

Every body in the universe attracts every other body with a gravitational force that decreases with distance as $1/r^2$.

But actually he knew more about the gravitational force: from the fact that bodies of different masses near the earth's surface accelerate downwards at the same rate, using $F = ma$ (his Second Law) if two bodies of different masses have the same acceleration they must be feeling forces in the same ratio as their masses (so a body twice as massive feels twice the gravitational force), that is, ***the gravitational force of attraction a body feels must be proportional to its mass***.

Now suppose we are considering the gravitational attraction between two bodies (as we always are), one of mass m_1, one of mass m_2. By Newton's Third Law, the force body 1 feels from 2 is equal in magnitude (but of course opposite in direction) to that 2 feels from 1. If we think of m_1 as the earth, the force m_2 feels is proportional to m_2, as argued above—so this must be true whatever m_1 is. And, since the situation is perfectly symmetrical, *the force must also be proportional to* m_1.

Putting all this together, the magnitude of the gravitational force between two bodies of masses m_1 and m_2 a distance r apart

$$\boldsymbol{F = Gm_1m_2/r^2.}$$

The constant $G = 6.67 \times 10^{-11}\ \mathrm{N.m^2/kg^2}$.

It is important to realize that G cannot be measured by any astronomical observations. For example, g at the surface of the earth is given by

$$g = Gm_E/r_E^2$$

where m_E is the mass and r_E the radius of the earth. Notice that by measuring g, and knowing r_E, we *can* find Gm_E. But this does *not* tell us what G is, since we don't know m_E! It turns out that this same problem arises with every astronomical observation. Timing the planets around the sun will give us Gm_{Sun}. So we *can* figure out the ratio of the sun's mass to the earth's, but we can't find an absolute value for either one.

The first measurement of G was made in 1798 by Cavendish, a century after Newton's work. Cavendish measured the tiny attractive force between lead spheres of known mass. For details on how an experiment at the University of Virginia in 1969 improved on Cavendish's work, click on the UVa Physics site here.

Cavendish said he was "*weighing the earth*" because once G is measured, he could immediately find the mass of the earth m_E from $g = Gm_E/r_E^2$, and then go on the find the mass of the sun, etc.

Describing the Solar System: Kepler's Laws

Newton's first clue that gravitation between bodies fell as the inverse-square of the distance may have come from comparing a falling apple to the falling moon, but important support for his idea was provided by a detailed description of planetary orbits constructed half a century earlier by Johannes Kepler.

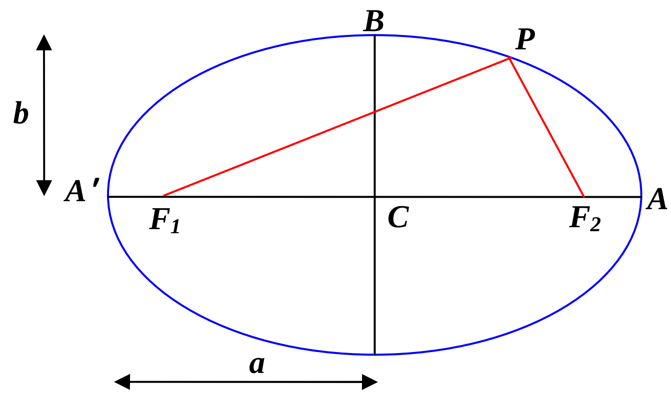
Kepler had inherited from Tycho Brahe a huge set of precise observations of planetary motions across the sky, spanning decades. Kepler himself spent eight years mathematically analyzing the observations of the motion of Mars, before realizing that Mars was moving in an *elliptical* path.

To appreciate fully how Kepler's discovery confirmed Newton's theory, it is worthwhile to review some basic properties of ellipses.

Mathematical Interlude: Ellipses 101

A *circle* can be defined as the set of all points which are the same distance R from a given point, so a circle of radius 1 centered at the origin O is the set of all points distance 1 from O.

An *ellipse* can be defined as the set of all points such that the *sum* of the distances from two fixed points is a constant length (which must obviously be greater than the distance between the two points!). This is sometimes called the gardener's definition: to set the outline of an elliptic flower bed in a lawn, a gardener would drive in two stakes, tie a loose rope between them, then pull the rope tight in all different directions to form the outline.



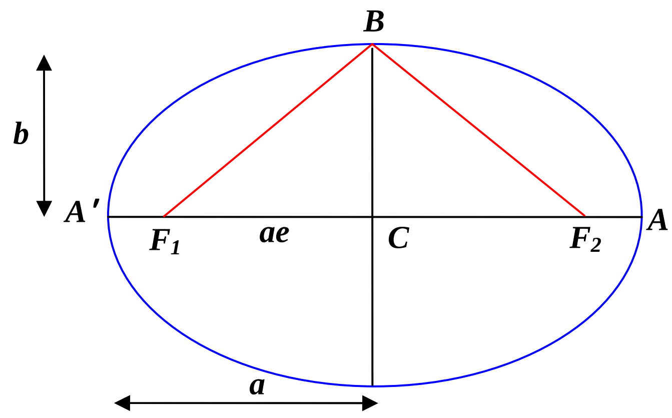
In the diagram, the stakes are at F_1, F_2, the **red lines** are the **rope**, P is an arbitrary point on the ellipse.

CA is called the semimajor axis length a, CB the semiminor axis, length b.

F_1, F_2 are called the foci (plural of focus).

Notice first that **the string has to be of length 2*a***, because it must stretch along the major axis from F_1 to A then back to F_2, and for that configuration there's a double length of string along F_2A and a single length from F_1 to F_2. But the length $A'F_1$ is the same as F_2A, so the total length of string is the same as the total length $A'A = 2a$.

Suppose now we put P at B. Since $F_1B = BF_2$, and the string has length $2a$, **the length $F_1B = a$**.



We get a useful result by applying Pythagoras' theorem to the triangle F_1BC,

$$(F_1C)^2 = a^2 - b^2.$$

(We shall use this shortly.)

Evidently, for a circle, $F_1C = 0$. The ***eccentricity*** of the ellipse is defined as the ratio of F_1C to a, so

$$\text{eccentricity } e = F_1C/a = \sqrt{1-(b/a)^2}.$$

The eccentricity of a circle is zero. The eccentricity of a long thin ellipse is just below one.

F_1 and F_2 on the diagram are called the *foci* of the ellipse (plural of *focus*) because if a point source of light is placed at F_1, and the ellipse is a mirror, it will reflect—and therefore *focus*—all the light to F_2. (This can be proved using the string construction.)

An ellipse is essentially a circle scaled shorter in one direction: in (x, y) coordinates it is described by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

a circle being given by $a = b$.

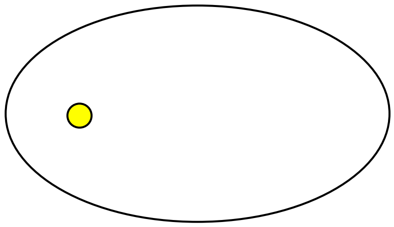
In fact, in analyzing planetary motion, it is more natural to take the origin of coordinates at the center of the Sun rather than the center of the elliptical orbit. It is also more convenient to take (r, θ) coordinates instead of (x, y) coordinates, because the strength of the gravitational force depends only on r. Therefore, the relevant equation describing a planetary orbit is the (r, θ) equation with the origin at one focus. For an ellipse of semi major axis a and eccentricity e the equation is:

$$\frac{a\left(1-e^2\right)}{r}=1+e\cos\theta.$$

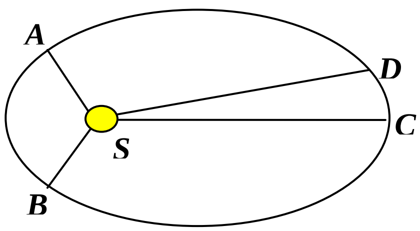
It is not difficult to prove that this is equivalent to the traditional equation in terms of x, y presented above.

Kepler summarized his findings about the solar system in his three laws:

1. The planets all move in elliptical orbits with the Sun at one focus.



2. As a planet moves in its orbit, the line from the center of the Sun to the center of the planet sweeps out equal areas in equal times, so if the area *SAB* (with curved side *AB*) equals the area *SCD*, the planet takes the same time to move from *A* to *B* as it does from *C* to *D*.



For my Flashlet illustrating this law, click here.

3. The time it takes a planet to make one complete orbit around the sun *T* (one planet year) is related to its average distance from the sun *R*:

$$T^2 \propto R^3.$$

In other words, if a table is made of the length of year *T* for each planet in the solar system, and its average distance from the sun *R*, and T^2/R^3 is computed for each planet, the numbers are all the same.

These laws of Kepler's are precise, but they are only *descriptive*—Kepler did not understand why the planets should behave in this way. Newton's great achievement was to prove that all this complicated behavior was the consequence of one simple law of attraction.

How Newton's Law of Universal Gravitation Explains Kepler's Laws

Surprisingly, the first of Kepler's laws—that the planetary paths are elliptical—is the toughest to prove beginning with Newton's assumption of inverse-square gravitation. Newton himself did it with an ingenious geometrical argument, famously difficult to follow. It can be more easily proved using calculus, but even this is nontrivial, and we shall not go through it in class. (The proof is given later in these notes, if you're curious to see how it's done.)

The best strategy turns out to be to attack the laws in reverse order.

Kepler's Third Law (well, for circular orbits)

It is easy to show how Kepler's Third Law follows from the inverse square law if we assume the planets move in perfect circles, which they *almost* do. The acceleration of a planet moving at speed v in a circular orbit of radius R is v^2/R towards the center. (Of course you already know this, but it is amusing to see how easy it is to prove using the Pythagoras diagram above: just replace the 5 meters by ½ at^2, the "horizontal" distance v by vt, write down Pythagoras' theorem and take the limit of a very small time.)

Newton's Second Law $F = ma$ for a planet in orbit becomes:

$$\frac{mv^2}{R} = G\frac{Mm}{R^2}.$$

Now the time for one orbit is $T = 2\pi R/v$, so dividing both sides of the equation above by R, we find:

$$\left(\frac{T}{2\pi}\right)^2 = \frac{R^3}{GM}, \quad \text{so} \quad \frac{T^2}{R^3} = \frac{4\pi^2}{GM}.$$

This is Kepler's Third Law: T^2/R^3 has the same numerical value for all the sun's planets.

Exercise: how are R, T related if the gravitational force is proportional to $1/R$? to $1/R^3$? To R?

The point of the exercise is that Kepler's Third Law, based on observation, forces us to the conclusion that the Law of Gravity is indeed inverse square.

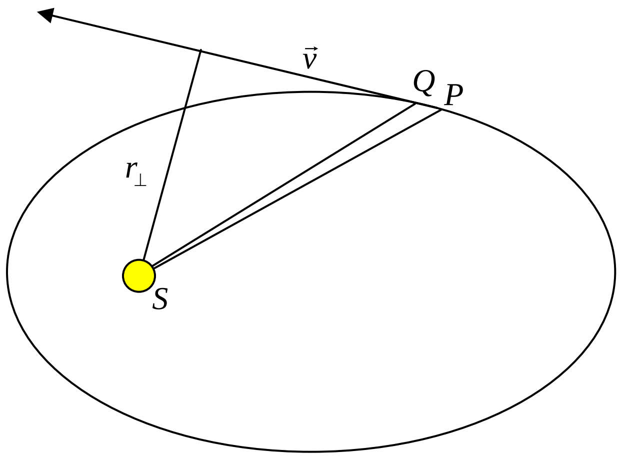
In fact, Newton went further—he generalized the proof to elliptic orbits, and established that for the inverse square law R must (for ellipses) be replaced by a, the semimajor axis of the ellipse, that is to say T^2/a^3 is the same for all planets. This is in fact exactly what Kepler found to be the case.

It follows immediately that all elliptic orbits with the ***same*** *major axis length,* ***whatever*** *their eccentricity, have the* ***same*** *orbital time.*

Kepler's Second Law

A planet in its path around the sun sweeps out equal areas in equal times.

Suppose at a given instant of time the planet is at point *P* in its orbit, moving with a velocity $\vec{v}$ meters per second in the direction along the tangent at *P* (see figure). In the next second it will move *v* meters, essentially along this line (the distance is of course greatly exaggerated in the figure) so the area swept out in that second is that of the triangle *SPQ*, where *S* is the center of the sun.



The area of triangle *SPQ* is just ½ base x height. The base *PQ* is *v* meters long, the height is the perpendicular distance $r_\perp$ from the vertex of the triangle at the sun *S* to the baseline *PQ*, which is just the tangential velocity vector $\vec{v}$.

Hence

$$\text{rate of sweeping out of area } = \frac{1}{2} r_\perp v .$$

Comparing this with the angular momentum *L* of the planet as it moves around the sun,

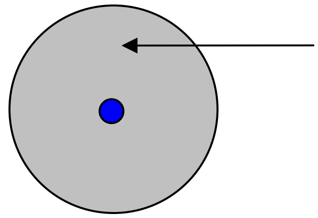
$$L = mvr_\perp$$

it becomes apparent that Kepler's Second Law, the constancy of the area sweeping rate, is telling us that *the angular momentum of the planet around the sun is constant*.

In fact,

$$\text{rate of sweeping out of area } = \frac{L}{2m}.$$

To see what this means, think of applying a force to a wheel on a fixed axle:



If the force is above the axle, as shown, the wheel will begin to turn anticlockwise, if it is below the axle the wheel will turn the other way—assuming no friction, the rate of change of angular momentum is equal to the torque, the product of the magnitude of the force and the perpendicular distance of the line of action of the force from the center of rotation, the axle. If the line of action of the force passes through the middle of the axle, there is no torque, no rotation, no change of angular momentum.

For the planet in orbit, the fact that the angular momentum about the sun does not change means that the force acting on the planet has no torque around the sun—the force is directly towards the sun. This now seems obvious, but Kepler himself thought the planets were pushed around their orbits by spokes radiating out from the sun. Newton realized that Kepler's Second Law showed this was wrong—the force must be directly towards the sun.

* Calculus Derivation of Kepler's First Law

Note: I'm including this derivation of the elliptic orbit just so you can see that it's calculus, not magic, that gives this result. This is an optional section, and will not appear on any exams.

We now back up to Kepler's *First* Law: proof that the orbit is in fact an ellipse if the gravitational force is inverse square. As usual, we begin with Newton's Second Law: $F = ma$, in vector form. The force is GMm/r^2 in a radial inward direction. But what is the radial acceleration? Is it just d^2r/dt^2? Well, no, because if the planet's moving in a circular orbit it's still accelerating inwards at $r\omega^2$ (same as v^2/r) even though r is not changing at all. The total acceleration is the *sum*, so $ma = F$ becomes:

$$\frac{d^2r}{dt^2} - r\omega^2 = -\frac{GM}{r^2}$$

This isn't ready to integrate yet, because ω varies too. But since the angular momentum L is constant, $L = mr^2\omega$, we can get rid of ω in the equation to give:

$$\begin{aligned}\frac{d^2r}{dt^2} &= -\frac{GM}{r^2} + r\left(\frac{L}{mr^2}\right)^2 \\ &= -\frac{GM}{r^2} + \frac{L^2}{mr^3}\end{aligned}$$

This equation can be integrated, using two very unobvious tricks, figured out by hindsight. The first is to change go from the variable *r* to its inverse, $u = 1/r$. The other is to use the constancy of angular momentum to change the variable *t* to θ.

Anyway,

$$L = mr^2\omega = \frac{m}{u^2}\frac{d\theta}{dt}$$

so

$$\frac{d}{dt} = \frac{Lu^2}{m}\frac{d}{d\theta}.$$

Therefore

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{L}{m}\frac{du}{d\theta}$$

and similarly

$$\frac{d^2r}{dt^2} = -\frac{L^2u^2}{m^2}\frac{d^2u}{d\theta^2}.$$

Substituting in the equation of motion gives:

$$\frac{d^2u}{d\theta^2} + u = \frac{GMm^2}{L^2}$$

This equation is easy to solve! The solution is

$$u = \frac{1}{r} = \frac{GMm^2}{L^2} + A\cos\theta$$

where *A* is a constant of integration, determined by the initial conditions.

This is equivalent to the standard (r,θ) equation of an ellipse of semi major axis *a* and eccentricity *e*, with the origin at one focus, which is:

$$\frac{a\left(1-e^2\right)}{r} = 1 + e\cos\theta.$$

Deriving Kepler's Third Law for Elliptical Orbits

The time it takes a planet to make one complete orbit around the Sun *T* (one planet year) is related to the semi-major axis *a* of its elliptic orbit by

$$T^2 \propto a^3.$$

We have already shown how this can be proved for circular orbits, however, since we have gone to the trouble of deriving the (r,θ) formula for an elliptic orbit, we add here the (optional) proof for that more general case.

(Note that this same result is derived in the next lecture using energy and angular momentum conservation—the proof given here is quicker, but depends on knowing the (r,θ) equation for the ellipse.)

The area of an ellipse is $\pi ab,$ and the rate of sweeping out of area is $L/2m$, so the time T for a complete orbit is evidently

$$T=\frac{\pi ab}{L/2m}.$$

Putting the equation

$$\frac{1}{r}=\frac{GMm^2}{L^2}+A\cos\theta$$

in the standard form

$$\frac{a\left(1-e^2\right)}{r}=1+e\cos\theta,$$

we find

$$L^2/GMm^2=a(1-e^2).$$

Now, the top point B of the semi-minor axis of the ellipse (see the diagram above) must be exactly a from F_1 (visualize the string F_1BF_2), so using Pythagoras' theorem for the triangle F_1OB we find

$$b^2=a^2\left(1-e^2\right).$$

Using the two equations above, the square of the orbital time

$$\begin{aligned}T^2&=\left(2m\pi ab\right)^2/L^2\\&=\left(2m\pi ab\right)^2/GMm^2a\left(1-e^2\right)\\&=\left(2m\pi ab\right)^2/GMm^2a\left(b^2/a^2\right)\\&=4\pi^2a^3/GM.\end{aligned}$$

We have established, then, that the time for one orbit depends *only* on the semimajor axis of the orbit: *it does not depend on how eccentric the orbit is.*

Visualizing Gravity: the Gravitational Field

Introduction

Let's begin with the *definition* of gravitational field:

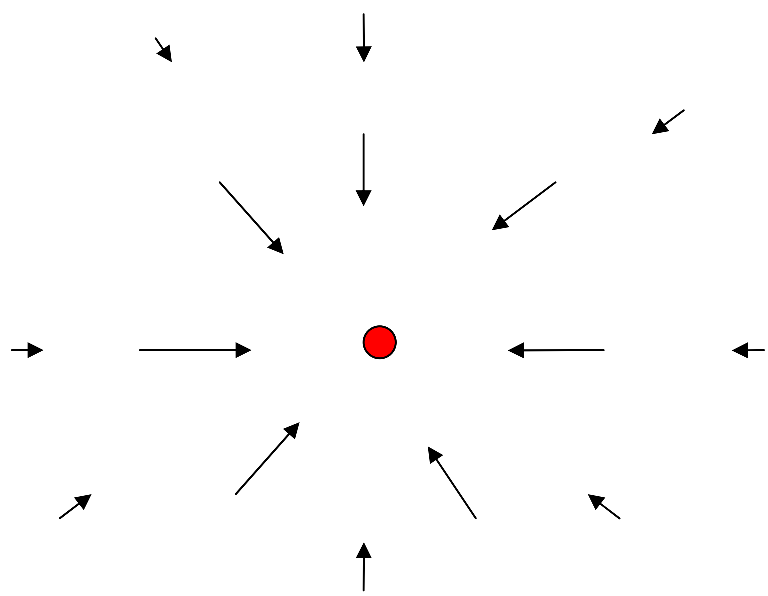
The gravitational field at any point P in space is defined as the gravitational force felt by a tiny unit mass placed at P.

So, to visualize the gravitational field, in this room or on a bigger scale such as the whole Solar System, imagine drawing a vector representing the gravitational force on a one kilogram mass at many different points in space, and seeing how the pattern of these vectors varies from one place to another (in the room, of course, they won't vary much!). We say "a tiny unit mass" because we don't want the gravitational field from the test mass itself to disturb the system. This is clearly not a problem in discussing planetary and solar gravity.

To build an intuition of what various gravitational fields look like, we'll examine a sequence of progressively more interesting systems, beginning with a simple point mass and working up to a hollow spherical shell, this last being what we need to understand the Earth's own gravitational field, both outside and inside the Earth.

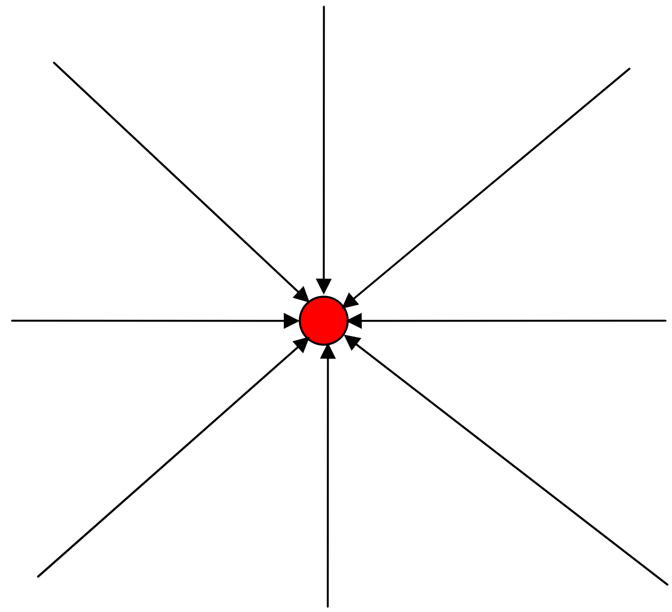
Field from a Single Point Mass

This is of course simple: we know this field has strength GM/r^2, and points towards the mass—the direction of the attraction. Let's draw it anyway, or, at least, let's draw in a few vectors showing its strength at various points:



This is a rather inadequate representation: there's a lot of blank space, and, besides, the field attracts in three dimensions, there should be vectors pointing at the mass in the air above (and below) the paper. But the picture does convey the general idea.

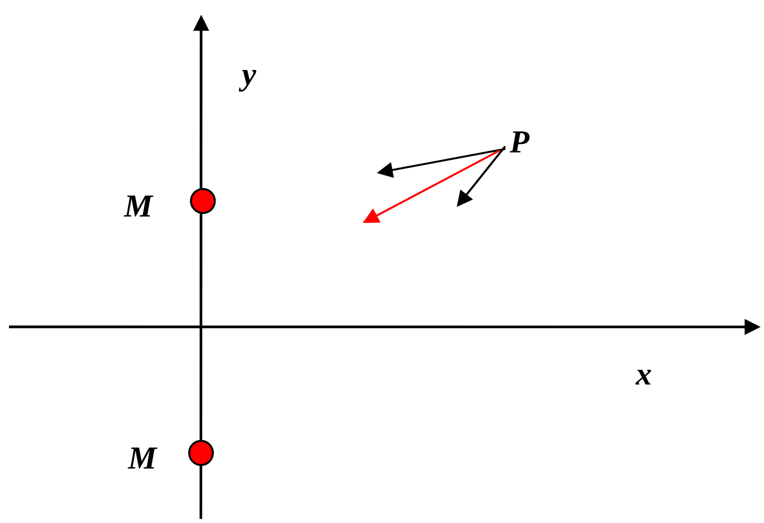
A different way to represent a field is to draw "field lines", curves such that at every point along the curve's length, its direction is the direction of the field at that point. Of course, for our single mass, the field lines add little insight:



The arrowheads indicate the direction of the force, which points the same way all along the field line. A shortcoming of the field lines picture is that although it can give a good general idea of the field, there is no precise indication of the field's *strength* at any point. However, as is evident in the diagram above, there *is* a clue: where the lines are closer together, the force is stronger. Obviously, we could put in a spoke-like field line anywhere, but if we want to give an indication of field strength, we'd have to have additional lines equally spaced around the mass.

Gravitational Field for Two Masses

The next simplest case is two equal masses. Let us place them symmetrically above and below the *x*-axis:



Recall Newton's Universal Law of Gravitation states that any two masses have a mutual gravitational attraction Gm_1m_2/r^2. A point mass m = 1 at P will therefore feel gravitational attraction towards both masses M, and a *total* gravitational field equal to the *vector sum of these two forces*, illustrated by the red arrow in the figure.

The Principle of Superposition

The fact that the total gravitational field is just given by adding the two vectors together is called the *Principle of Superposition*. This may sound really obvious, but in fact it isn't true for every force found in physics: the strong forces between elementary particles don't obey this principle, neither do the strong gravitational fields near black holes. But just adding the forces as vectors works fine for gravity almost everywhere away from black holes, and, as you will find later, for electric and magnetic fields too. Finally, superposition works for any number of masses, not just two: the total gravitational field is the vector sum of the gravitational fields from all the individual masses. Newton used this to prove that the gravitational field outside a solid sphere was the same as if all the mass were at the center by imagining the solid sphere to be composed of many small masses—in effect, doing an integral, as we shall discuss in detail later. He also invoked superposition in calculating the orbit of the Moon precisely, taking into account gravity from both the Earth and the Sun.

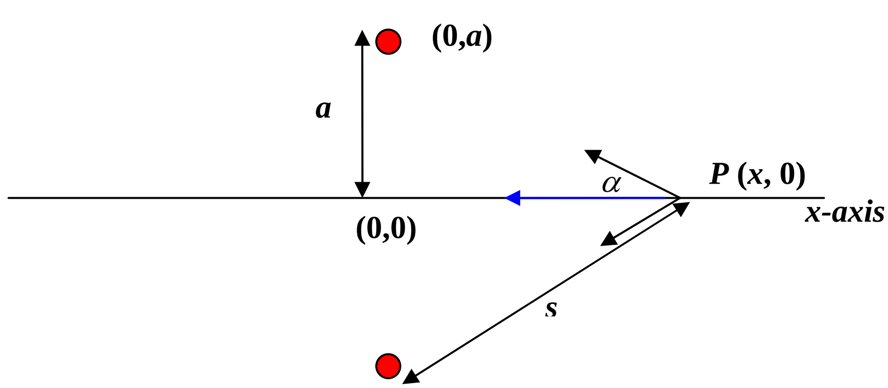
Exercise: For the two mass case above, sketch the gravitational field vector at some other points: look first on the x-axis, then away from it. What do the *field lines* look like for this two mass case? Sketch them *in the neighborhood of the origin*.

Field Strength at a Point Equidistant from the Two Masses

It is not difficult to find an exact expression for the gravitational field strength from the two equal masses at an equidistant point P.

Choose the x,y axes so that the masses lie on the y-axis at (0, a) and (0,-a).

By symmetry, the field at P must point *along* the x-axis, so all we have to do is compute the strength of the x-component of the gravitational force from one mass, and double it.



If the distance from the point P to one of the masses is s, the gravitational force towards that mass has strength GM/s^2. This force has a component along the x-axis equal to $\left(GM/s^2\right)\cos\alpha$, where α is the angle between the line from P to the mass and the x-axis, so the total gravitational force on a small unit mass at P is $2\left(GM/s^2\right)\cos\alpha$ directed along the x-axis.

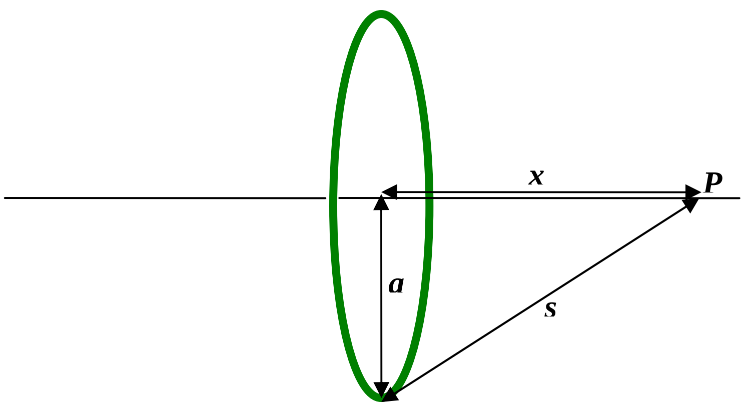
From the diagram, $\cos\alpha = x/s$, so the force on a unit mass at P from the two masses M is

$$F = -\frac{2GMx}{\left(x^2+a^2\right)^{3/2}}$$

in the x-direction. Note that the force is exactly zero at the origin, and everywhere else it points towards the origin.

Gravitational Field from a Ring of Mass

Now, as long as we look *only* on the x-axis, this identical formula works for a *ring* of mass $2M$ in the y, z plane! It's just a three-dimensional version of the argument above, and can be visualized by rotating the two-mass diagram above around the x-axis, to give a ring perpendicular to the paper, or by imagining the ring as made up of many beads, and taking the beads in pairs opposite each other.



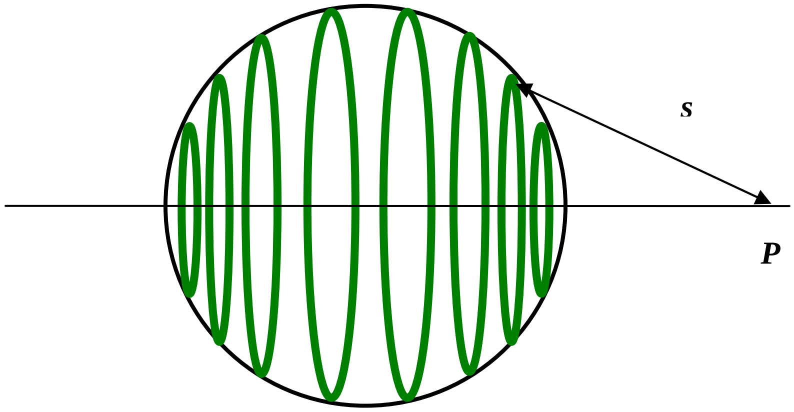
Bottom line: the field from a ring of *total* mass M, radius a, at a point P on the axis of the ring distance x from the center of the ring is

$$F = -\frac{GMx}{\left(x^2+a^2\right)^{3/2}}.$$

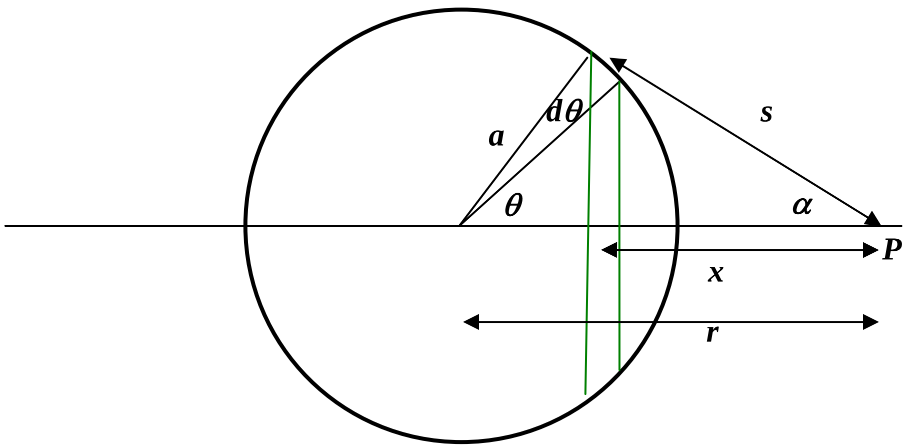
*Field Outside a Massive Spherical Shell

This is an optional section: you can safely skip to the result on the last line. In fact, you will learn an easy way to derive this result using Gauss's Theorem when you do Electricity and Magnetism. I just put this section in so you can see that this result can *be derived by the straightforward, but quite challenging, method of adding the individual gravitational attractions from all the bits making up the spherical shell.*

What about the gravitational field from a hollow spherical *shell* of matter? Such a shell can be envisioned as *a stack of rings*.



To find the gravitational field at the point *P*, we just add the contributions from all the rings in the stack.



In other words, we divide the spherical shell into narrow "zones": imagine chopping an orange into circular slices by parallel cuts, perpendicular to the axis—but of course our shell is just the *skin* of the orange! One such slice gives a ring of skin, corresponding to the surface area between two latitudes, the two parallel lines in the diagram above. Notice from the diagram that this "ring of skin" will have radius $a\sin\theta$, therefore circumference $2\pi a\sin\theta$ and breadth $ad\theta$, where we're taking $d\theta$ to be *very small*. This means that the *area of the ring* of skin is

$$\text{length}\times\text{breadth} \;=2\pi a\sin\theta\times ad\theta\,.$$

So, if the shell has mass ρ per unit area, this ring has mass $2\pi a^2\rho\sin\theta d\theta$, and the gravitational force at *P* from this ring will be

$$F(\text{ring } d\theta) = -\frac{2Gx\pi a^2 \rho \sin\theta d\theta}{\left(x^2 + a^2\right)^{3/2}}.$$

Now, to find the total gravitational force at P from the *entire shell* we have to add the contributions from each of these "rings" which, taken together, make up the shell. In other words, we have to *integrate* the above expression in θ from $\theta = 0$ to $\theta = \pi$.

So the gravitational field is:

$$F = -\int_0^\pi \frac{2Gx\pi a^2 \rho \sin\theta d\theta}{\left(x^2 + a^2\right)^{3/2}} = -\int_0^\pi \frac{2Gx\pi a^2 \rho \sin\theta d\theta}{s^3}.$$

In fact, this is quite a tricky integral: θ, x and s are all varying! It turns out to be is easiest done by switching variables from θ to s.

Label the distance from P to the center of the sphere by r. Then, from the diagram, $s^2 = r^2 + a^2 - 2ar\cos\theta$, and a, r are constants, so $sds = ar\sin\theta d\theta$,

$$\text{and } F = -\int_0^\pi \frac{2Gx\pi a^2 \rho \sin\theta d\theta}{s^3} = -\int_{r-a}^{r+a} \frac{2Gx\pi a^2 \rho}{s^3} \cdot \frac{sds}{ar} = -\frac{2Ga^2 \rho \pi}{ar} \int_{r-a}^{r+a} \frac{xds}{s^2}.$$

Now $x = s\cos\alpha$, and from the diagram $a^2 = s^2 + r^2 - 2sr\cos\alpha$, so $x = \dfrac{s^2 + r^2 - a^2}{2r}$,

and, writing $4\pi a^2 \rho = M$,

$$F = \frac{GM}{4ar^2} \int_{r-a}^{r+a} \left(1 + \frac{r^2 - a^2}{s^2}\right) ds = \frac{GM}{4ar^2}\left(2a + \left(r^2 - a^2\right)\left(\frac{1}{r-a} - \frac{1}{r+a}\right)\right) = \frac{GM}{r^2}.$$

The derivation was rather lengthy, but the answer is simple:

The gravitational field outside a uniform spherical shell is *GM/r^2* towards the center.

And, there's a bonus: for the ring, we only found the field *along the axis*, but for the spherical shell, once we've found it in one direction, *the whole problem is solved*—for the spherical shell, the field must be the same in all directions.

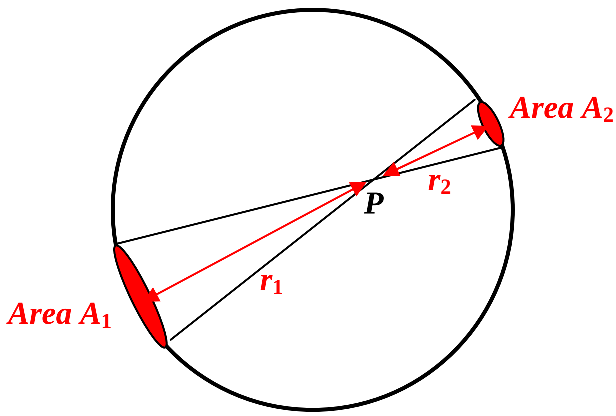
Field Outside a Solid Sphere

Once we know the gravitational field outside a shell of matter is the same as if all the mass were at a point at the center, it's easy to find the field outside a solid sphere: that's just a nesting set of shells, like spherical Russian dolls. Adding them up,

The gravitational field outside a uniform sphere is GM/r^2 towards the center.

There's an added bonus: since we found this result be adding uniform spherical shells, it is *still true* if the shells have *different densities*, provided the density of each shell is the same in all directions. The inner shells could be much denser than the outer ones—as in fact is the case for the Earth.

Field *Inside* a Spherical Shell



This turns out to be surprisingly simple! We imagine the shell to be very thin, with a mass density ρ kg per square meter of surface. Begin by drawing a two-way cone radiating out from the point P, so that it includes two small areas of the shell on opposite sides: these two areas will exert gravitational attraction on a mass at P in opposite directions. It turns out that they exactly *cancel*.

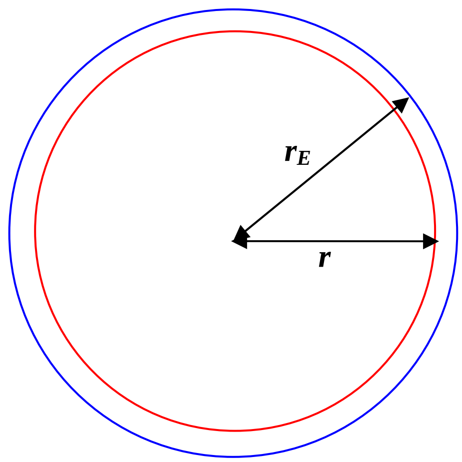
This is because the ratio of the areas A_1 and A_2 at distances r_1 and r_2 are given by $A_1 / A_2 = r_1^2 / r_2^2$: since the cones have the same angle, if one cone has twice the height of the other, its base will have twice the diameter, and therefore *four times the area*. Since the masses of the bits of the shell are proportional to the areas, the ratio of the masses of the cone bases is also r_1^2 / r_2^2. But the gravitational attraction at P from these masses goes as Gm / r^2, and that r^2 term *cancels* the one in the areas, so the two opposite areas have equal and opposite gravitational forces at P.

In fact, the gravitational pull from every small part of the shell is balanced by a part on the opposite side—you just have to construct a lot of cones going through P to see this. (There is one slightly tricky point—the line from P to the sphere's surface will in general cut the surface at an angle. However, it will cut the opposite bit of sphere *at the same angle*, because any line passing through a sphere hits the two surfaces at the same angle, so the effects balance, and the base areas of the two opposite small cones are *still* in the ratio of the squares of the distances r_1, r_2.)

Field Inside a Sphere: How Does *g* Vary on Going Down a Mine?

This is a practical application of the results for shells. On going down a mine, if we imagine the Earth to be made up of shells, we will be inside a shell of thickness equal to the depth of the mine, so will feel *no* net gravity from that part of the Earth. However, we will be *closer* to the remaining shells, so the force from them will be intensified.

Suppose we descend from the Earth's radius r_E to a point distance r from the center of the Earth. What fraction of the Earth's mass is still attracting us towards the center? Let's make life simple for now and assume the Earth's density is *uniform*, call it ρ kg per cubic meter.



Then the fraction of the Earth's mass that is still attracting us (because it's closer to the center than we are—inside the red sphere in the diagram) is $V_{red}/V_{blue} = \frac{4}{3}\pi r^3 / \frac{4}{3}\pi {r_E}^3 = r^3/r_E^3$.

The gravitational attraction from this mass at the bottom of the mine, distance r from the center of the Earth, is proportional to mass/r^2. We have just seen that the mass is itself proportional to r^3, so the actual gravitational force felt must be proportional to $r^3/r^2 = r$.

That is to say, the gravitational force on going down inside the Earth is *linearly proportional to distance from the center*. Since we already know that the gravitational force on a mass m at the Earth's surface $r = r_E$ is mg, it follows immediately that in the mine the gravitational force must be

$$F = mgr/r_E .$$

So there's no force at all at the center of the Earth—as we would expect, the masses are attracting equally in all directions.

Working with Gravity: Potential Energy

Gravitational Potential Energy near the Earth

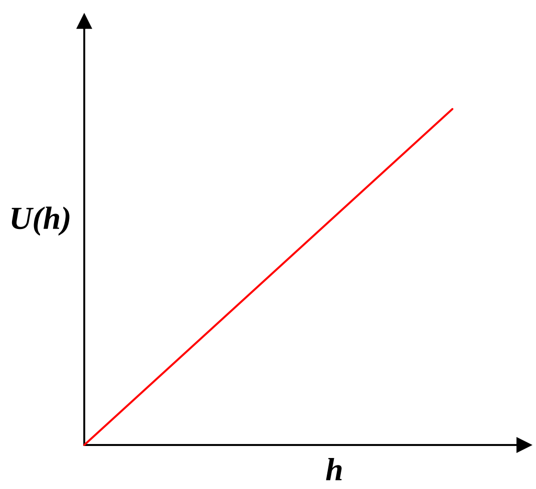
We first briefly review the familiar subject of gravitational potential energy near the Earth's surface, such as in a room. The gravitational force is of course $\vec{F} = m\vec{g}$ vertically downwards.

To raise a mass *m*, we must apply an upward force $-\vec{F}$, balancing gravity, so the net force on the body is zero and it can move upwards at a steady speed (ignoring air resistance, of course, and assuming we gave it a tiny extra push to get it going).

Applying the steady force $-\vec{F}$ as the mass moves a small distance $\Delta\vec{r}$ takes work $-\vec{F}\cdot\Delta\vec{r}$, and to raise the mass *m* through a height *h* takes work ***mgh***. This energy is stored and then, when the object falls, released as kinetic energy. For this reason it is called *potential energy*, being "potential kinetic energy", and written

$$U = U(h) = mgh.$$

Note one obvious ambiguity in the definition of potential energy: do we measure *h* from the floor, from the top of our workbench, or what? That depends on how far we will allow the raised object to fall and convert its potential energy to kinetic energy—but the main point is *it doesn't matter where the zero is set*, the quantity of physical interest is always a *difference* of potential energies between two heights—that's how much kinetic energy is released when it falls from one height to the other. (Perhaps we should mention that some of this potential energy may go to another form of energy when the object falls—if there is substantial air resistance, for example, some could end up eventually as heat. We shall ignore that possibility for now.)



Onward and Upward

Let's now consider the work involved in lifting something so high that the Earth's gravitational pull becomes noticeably weaker.

It will still be true that lifting through $\Delta\vec{r}$ takes work $-\vec{F}\cdot\Delta\vec{r}$, but now $\vec{F}(\vec{r}) = GMm/r^2$, downwards. So

$$dU = -\vec{F}\cdot d\vec{r} = \frac{GMm}{r^2}dr$$

and to find the total work needed to lift a mass m from the Earth's surface (r_E from the center of the Earth) to a point distance r from the center we need to do an integral:

$$U(r) - U(r_E) = \int_{r_E}^{r} \frac{GMm}{r'^2}dr' = GMm\left(\frac{1}{r_E} - \frac{1}{r}\right).$$

First check that this makes sense close to the Earth's surface, that is, in a room. For this case,

$$r = r_E + h, \quad \text{where } h \ll r_E.$$

Therefore

$$\begin{aligned} U(r) - U(r_E) &= GMm\left(\frac{1}{r_E} - \frac{1}{r}\right) \\ &= GMm\left(\frac{r_E + h - r_E}{r_E(r_E + h)}\right) \\ &\cong GMm\left(\frac{h}{r_E^2}\right) \\ &= mgh \end{aligned}$$

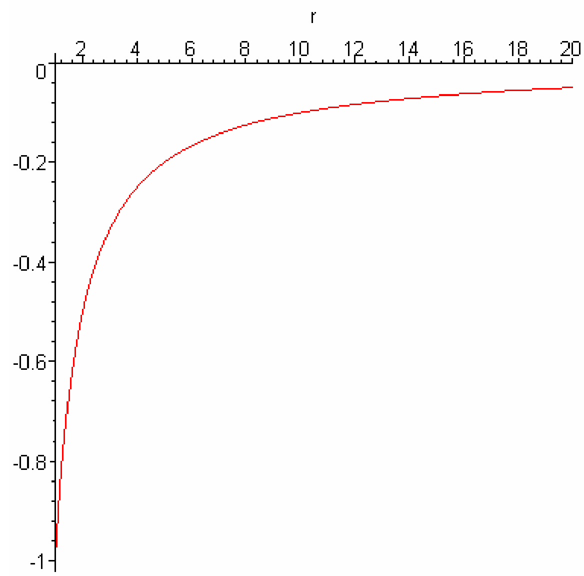
where the only approximation is to replace $r_E + h$ by r_E in the denominator, giving an error of order h/r_E, parts per million for an ordinary room.

To see what this potential function looks like on a larger scale, going far from the Earth, it is necessary first to decide where it is most natural to set it equal to zero. The standard convention is to set the potential energy equal to zero at r = infinity! The reason is that if two bodies are very far from each other, they have no influence on each other's movements, so it is pointless to include a term in their total energy which depends on their mutual interaction.

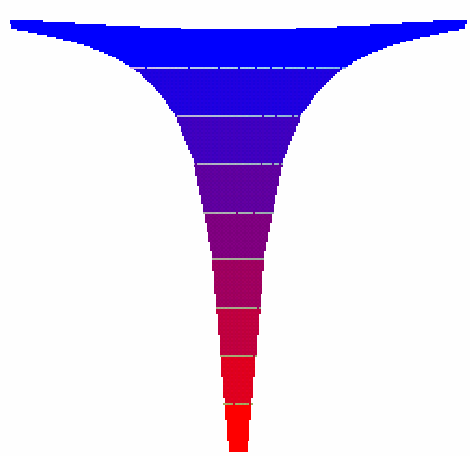
Taking the potential energy zero at infinity gives the simple form

$$U(r) = -\frac{GMm}{r},$$

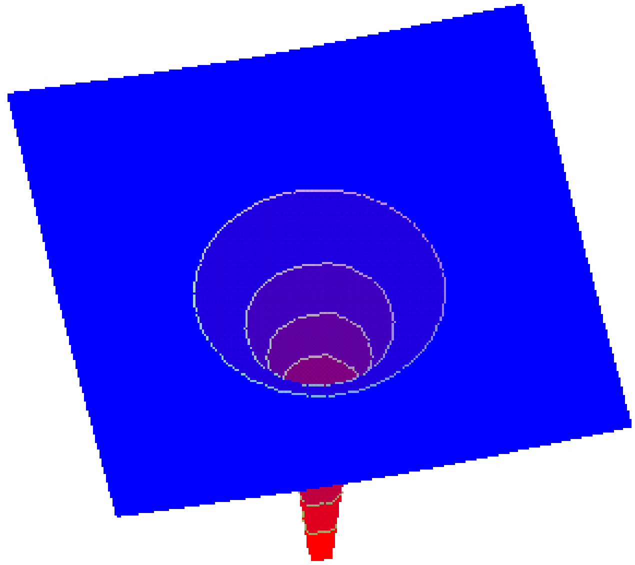
we plot it below with r in units of Earth radii. The energy units are GMm/r_E, the −1 at the far left being at the Earth's surface (r = 1), and the first steep almost linear part corresponds to mgh.



The above is a map of the potential energy "hill" to be climbed in going away from the Earth vertically upwards from any point. To gain something closer to a three-dimensional perspective, the Earth can be visualized as being at the bottom of a "potential well" with flared sides, like this:



Or, from a different perspective:



Of course, this is still only in two dimensions, but that's fine for most gravitational problems: planetary orbits are only two-dimensional. A satellite in a circular orbit around the Earth can be imagined as a frictionless particle sliding around inside this "cone" at a fixed height, for an elliptic orbit the particle would slide between different heights.

Gravitational Potential Energy in a Two Body System

By this, I mean how do we extend the above picture of gravitational potential as a "well" going down out of a flat plane to, for example, the combined potential energies of a mass in the gravitational fields of both the Earth and the Moon, as would occur on a flight to the Moon.

From the beginning of the previous section, the potential energy difference between any two points from the gravitational force of a single body is the work done against that force in going from one point to the other,

$$U(\vec{r}_2) - U(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}.$$

It doesn't matter how the path gets from $\vec{r}_1$ to $\vec{r}_2$: if it took different amounts of work depending on the path, we could gain energy by having a mass go up one path and down the other, a perpetual motion machine. The fact that this is not true means the gravitational field is *conservative*: gravitational potential energy can b a term in a conservation of energy equation.

Recall from the previous lecture that the gravitational field obeys the Law of Superposition: to find the total gravitational force on a mass from the gravitational field of both the Earth and the Moon, we just add the vectors representing the separate forces. It follows immediately from this that, putting $\vec{F} = \vec{F}_{\text{Earth}} + \vec{F}_{\text{Moon}}$, the gravitational potential energy difference between two points is simply the sum of the two terms.

From this, then, the potential energy of a mass somewhere between the Earth and the Moon is

$$U_{\text{total}}(\vec{r}) = -\frac{GM_E m}{|\vec{r}-\vec{r}_{CE}|} - \frac{GM_M m}{|\vec{r}-\vec{r}_{CM}|}$$

taking as usual $U(\infty)=0$, and $\vec{r}_{CE}, \vec{r}_{CM}$ are the coordinates of the centers of the Earth and the Moon respectively.

It's worth visualizing this combined potential: it would look like two of these cone-like wells, one much smaller than the other, in what is almost a plain. Going in a straight line from inside one well to the inside of the other would be uphill then downhill, and at the high point of the journey the potential energy would be flat, meaning that the gravitational pull of the Earth just cancels that of the Moon, so no work is being done in moving along the line at that point. The *total* potential energy there is still of course negative, that is, below the value (zero) far away in the plain.

Gravitational Potential

The *gravitational potential* is defined as the gravitational *potential energy per unit mass*, and is often written $\varphi(\vec{r})$. We shall rarely use it—the problems we encounter involve the potential energy of a given mass *m*. (But $\varphi(\vec{r})$ is a valuable concept in more advanced treatments. It is analogous to the electrostatic potential, and away from masses obeys the same partial differential equation, $\nabla^2\varphi(\vec{r})=0$.)

Escape!

How fast must a rocket be moving as it escapes the atmosphere for it to escape entirely from the Earth's gravitational field? This is the famous escape velocity, and, neglecting the depth of the atmosphere, it clearly needs sufficient initial kinetic energy to climb all the way up the hill,

$$\frac{1}{2}mv_{escape}^2 = \frac{GMm}{r_E}, \quad v_{escape} = \sqrt{\frac{2GM}{r_E}}.$$

This works out to be about 11.2 km per sec. For the Moon, escape velocity is only 2.3 km per second, and this is the reason the Moon has no atmosphere: if it had one initially, the Sun's heat would have been sufficient to give the molecules enough thermal kinetic energy to escape. In an atmosphere in thermal equilibrium, all the molecules have on average the same kinetic energy. This means lighter molecules on average move faster. On Earth, any hydrogen or helium in the atmosphere would eventually escape for the same reason.

Exercise: Saturn's moon Titan is the same size as our Moon, but Titan has a thick atmosphere. Why?

Exercise: Imagine a tunnel bored straight through the Earth emerging at the opposite side of the globe. The gravitational force in the tunnel is $F = mgr/r_E$, as derived above.

(a) Find an expression for the gravitational potential energy in the tunnel. Take it to be zero at the center of the Earth.

(b) Now sketch a graph of the potential energy as a function of distance from the Earth's center, beginning at the center but continuing beyond the Earth's radius to a point far away. This curve must be continuous. Conventionally, the potential energy is defined by requiring it to be zero at infinity. How would you adjust your answer to give this result?

Potential and Kinetic Energy in a Circular Orbit

The equation of motion for a satellite in a circular orbit is

$$\frac{mv^2}{r} = \frac{GMm}{r^2}.$$

It follows immediately that the kinetic energy

$$K.E. = \tfrac{1}{2}mv^2 = \tfrac{1}{2}GMm/r = -\tfrac{1}{2}U(r),$$

that is, the Kinetic Energy = –1/2 (Potential Energy) so *the total energy in a circular orbit is half the potential energy*.

The satellite's motion can be visualized as circling around trapped in the circular potential "well" pictured above. How fast does move? It is easy to check that for this circular orbit

$$v_{orbit} = \sqrt{\frac{GM}{r_{orbit}}}.$$

Recalling that the escape velocity from this orbit is $v_{escape} = \sqrt{2GM/r_{orbit}}$, we have

$$v_{escape} = \sqrt{2}v_{orbit}$$

relating speed in a circular planetary orbit to the speed necessary, starting at that orbit, to escape completely from the sun's gravitational field.

This result isn't surprising: increasing the speed by $\sqrt{2}$ *doubles* the kinetic energy, which would then *exactly equal* the potential energy: that means just enough kinetic energy for the satellite to climb the hill completely out of the "well".

Bottom line: the *total* energy of a planet of mass *m* in a circular orbit of radius *r* about a Sun of mass *M* is

$$E_{tot} = -\frac{GMm}{2r}.$$

Elliptic Orbits: Paths to the Planets

Deriving Essential Properties of Elliptic Orbits

From a practical point of view, elliptical orbits are a lot more important than circular orbits. A spaceship leaving earth and going in a circular orbit won't get very far. And although proving the planetary orbits *are* elliptical is quite a tricky exercise (the details can be found in the last section of the *Discovering Gravity* lecture), once that is established a lot can be deduced without further fancy mathematics.

Think about an astronaut planning a voyage from earth to Mars. The two important questions (apart from can I get back?) are:

How much fuel will this trip need?
How much time will it take?

It is crucial to minimize the fuel requirement, because lifting fuel into orbit is extremely expensive.

Ignoring minor refinements like midcourse corrections, the spaceship's trajectory to Mars will be along an elliptical path. We can calculate the amount of fuel required if we know the *total energy* of the ship in this elliptical path, and we can calculate the time needed if we know the *orbital time* in the elliptical path because, as will become apparent, following the most fuel-efficient path will take the ship exactly half way round the ellipse.

Remarkably, for a spaceship (or a planet) in an elliptical orbit, *both the total energy and the orbital time depend only on the length of the major axis of the ellipse*—as we shall soon show. Visualizing the orbit of the spaceship going to Mars, and remembering it is an ellipse with the sun at one focus, the smallest ellipse we can manage has the point furthest from the sun at Mars, and the point nearest to the sun at earth.

(*Important Exercise*: Sketch the orbits of earth and Mars, and this elliptical trajectory.)

This then immediately gives us the major axis of this smallest ellipse, so we can figure out, from the results given below, how much fuel and time this will take.

Here are the two basic relevant facts about elliptical orbits:

1. The time to go around an elliptical orbit once depends only on the length *a* of the semimajor axis, not on the length of the minor axis:

$$T^2 = \frac{4\pi^2 a^3}{GM}.$$

2. The total energy of a planet in an elliptical orbit depends only on the length *a* of the semimajor axis, not on the length of the minor axis:

$$E_{tot} = -\frac{GMm}{2a}.$$

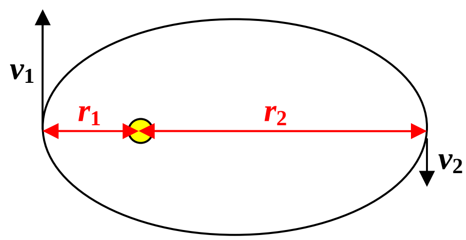
These results will get you a long way in understanding the orbits of planets, asteroids, spaceships and so on—and, given that the orbits are elliptical, they are fairly easy to prove.

They follow from the two *conservation laws*:

total energy stays constant
angular momentum stays constant

throughout the elliptical orbital motion.

We'll derive the results for a planet, beginning with the conservation laws. In fact, it turns out that all we need to use is *that the energy and angular momentum are the same at the two extreme points of the orbit*:



Labeling the distance of closest approach r_1, and the speed at that point v_1, the furthest point r_2, the speed there v_2, we have

$$mv_1r_1 = mv_2r_2 = L$$

and

$$\tfrac{1}{2}mv_1{}^2 - GMm/r_1 = \tfrac{1}{2}mv_2{}^2 - GMm/r_2 = E.$$

From the second equation,

$$\tfrac{1}{2}mv_1{}^2 - GMm/r_1 - (\tfrac{1}{2}mv_2{}^2 - GMm/r_2) = 0.$$

Rearranging, and dropping the common factor m,

$$\tfrac{1}{2}\left(v_1^2 - v_2^2\right) = GM\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Using the angular momentum equation to write $v_1 = L/mr_1$, $v_2 = L/mr_2$, and substituting these values in this equation gives L^2 in terms of r_1, r_2:

$$\frac{L^2}{2m^2}\left(\frac{1}{r_1^2}-\frac{1}{r_2^2}\right)=\frac{L^2}{2m^2}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)=GM\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

from which

$$\frac{L^2}{2m^2}=\frac{GM}{\left(\frac{1}{r_1}+\frac{1}{r_2}\right)}.$$

The area of the ellipse is πab (recall it's a circle squashed by a factor b/a in one direction, so πa^2 becomes πab), and the rate of sweeping out of area is $L/2m$, so the time T for a complete orbit is given by:

$$T^2=\frac{(\pi ab)^2}{L^2/4m^2}=\frac{2(\pi ab)^2}{GM}\left(\frac{1}{r_1}+\frac{1}{r_2}\right).$$

To make further progress in proving the orbital time T depends on a but not on b, we need to express r_1, r_2 in terms of a and b.

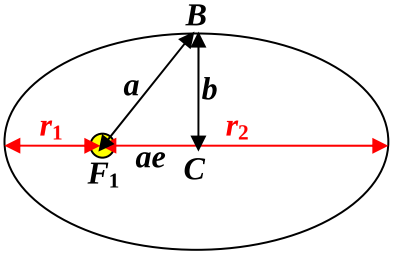
Useful Ellipse Factoid

Recall that the sun is at a focus F_1 of the elliptical path (see figure below), and (from the "string" definition of the ellipse) the distance from the sun to point B at the end of the minor axis is a. Pythagoras' theorem applied to the triangle F_1BC gives

$$a^2\left(1-e^2\right)=b^2$$

and from the figure

$$r_1=a(1-e)$$
$$r_2=a(1+e)$$



Therefore

$$r_1r_2=b^2.$$

Also from the figure

$$\frac{r_1 + r_2}{2} = a,$$

so we have the amusing result that

the semimajor axis *a* is the *arithmetic* mean of r_1, r_2 and the semiminor axis *b* is their *geometric* mean, and furthermore

$$\left(\frac{1}{r_1} + \frac{1}{r_2}\right) = \frac{r_1 + r_2}{r_1 r_2} = \frac{2a}{b^2}.$$

Deriving Kepler's Law

We can immediately use the above result to express the angular momentum L very simply:

$$\frac{L^2}{2m^2} = \frac{GM}{\left(\frac{1}{r_1} + \frac{1}{r_2}\right)} = \frac{GM\,b^2}{2a}.$$

We're now ready to find the time for one orbit T. Remember T is the total area of the orbit divided by the rate area is swept out, and that rate is $L/2m$, so:

$$T^2 = \frac{(\pi ab)^2}{L^2/4m^2} = \frac{2(\pi ab)^2\, 2a}{GMb^2} = \frac{4\pi^2 a^3}{GM}.$$

That is,

$$T^2 = \frac{4\pi^2 a^3}{GM},$$

a simple generalization of the result for circular orbits.

To prove that the total energy only depends on the length of the major axis, we simply add the total energies at the two extreme points:

$$\tfrac{1}{2} m {v_1}^2 - GMm/r_1 + \tfrac{1}{2} m {v_2}^2 - GMm/r_2 = 2E.$$

The substitution $v_1 = L/mr_1$, $v_2 = L/mr_2$ in this equation gives

$$\frac{L^2}{2m}\left(\frac{1}{{r_1}^2} + \frac{1}{{r_2}^2}\right) - GMm\left(\frac{1}{r_1} + \frac{1}{r_2}\right) = 2E.$$

Writing $\left(\frac{1}{r_1^{\;2}}+\frac{1}{r_2^{\;2}}\right)=\left(\frac{1}{r_1}+\frac{1}{r_2}\right)^2-\frac{2}{r_1r_2}$, it is easy to check that

$$E=-\frac{GM}{r_1+r_2}=-\frac{GM}{2a}$$

where a is the semimajor axis.

Exercise: From $\frac{L^2}{2m^2}=\frac{GMb^2}{2a}$, find the speed of the planet at it goes through the point B at the end of the minor axis. What is its potential energy at that point? Deduce that the total energy depends only on the length of the major axis. (This is an alternative derivation.)

Hyperbolic (and Parabolic?) Orbits

Imagining the satellite as a particle sliding around in a frictionless well representing the potential energy as pictured above, one can see how both circular and elliptical orbits might occur.

(*Optional*: More formally, we solved the equation of motion in earlier notes to find

$$\frac{1}{r}=\frac{GMm^2}{L^2}+A\cos\theta$$

which is equivalent to the equation for an ellipse

$$\frac{a\left(1-e^2\right)}{r}=1+e\cos\theta$$

as discussed there.)

However, that is not the whole story: what if a rogue planet comes flying towards the Solar System from outer space? What kind of orbit will it follow as it encounters the Sun's gravity? In fact, our analysis of the equations of motion is equally valid in this case, and the (r,θ) equation is the same as that above! The new wrinkle is that e, which is always less than one for an ellipse, becomes greater than one, and this means that for some angles r can be infinite (the right-hand side of the above equation can be zero). The orbit is a *hyperbola*: the rogue comes in almost along a straight line at large distances, the Sun's gravity causes it to deviate, it swings around the Sun, then recedes tending to another straight line path as it leaves the System.

There is also the theoretical possibility of a parabolic orbit, going out to infinity but never approaching a straight line asymptote. However, this requires *exactly* the correct energy—the slightest difference would turn it into a very long ellipse or a hyperbola. In practice, of course, this delicate energy tuning would be upset by gravitational attraction from other planets.

Really Getting Out There: the Slingshot

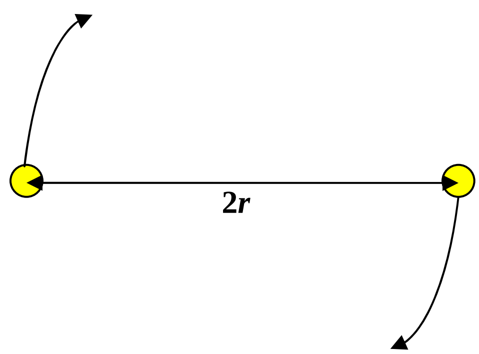
Although the elliptic orbit touching the (approximately) circular orbits of earth and Mars is the most economical orbit of getting to Mars, trips to the outer planets can get help. Imagine a slowly moving spaceship reaching Jupiter's orbit at a point some distance in front of Jupiter as Jupiter moves along the orbit. In Jupiter's frame of reference, this ship is moving towards Jupiter at a speed roughly equal to Jupiter's own speed relative to the sun. In Jupiter's frame, assuming the spaceship is sufficiently far from the orbit that it doesn't crash into Jupiter, it will fall towards Jupiter, swing around the back, and then be flung forward. In the sun's frame, the gravitational pull on the spaceship from Jupiter was strongest as the spaceship swung behind Jupiter, and this pull accelerated the spaceship in the same direction Jupiter moves in the orbit, so the spaceship subsequently moves ahead of Jupiter, having gained enough energy to move further out in the solar system. This is at the expense of Jupiter: during the time the spaceship was swing behind Jupiter, it slowed Jupiter's orbital speed—but not much! The slingshot is obviously a delicate operation: you don't want to crash into Jupiter, but also you don't want to be trapped in an elliptic orbit around Jupiter. But the benefits are so great that in practice all spaceships venturing to the outer planets use it, often more than once.

To practice the slingshot yourself, check out the flashlet here.

Binary Stars and Tidal Forces

Binary Stars

Up to this point, we've been considering gravitational attraction between pairs of objects where one of them was much heavier than the other, and was taken to be fixed. That is an excellent approximation for the Sun and the planets, or the planets and their satellites, but is not perfect. To see where it really breaks down, consider a binary star system with two equally massive stars. (Binary star systems are quite common, in fact most stars are in them.) In the simplest case, the two stars will orbit each other in circles, or, rather, by symmetry they will orbit a common central point:

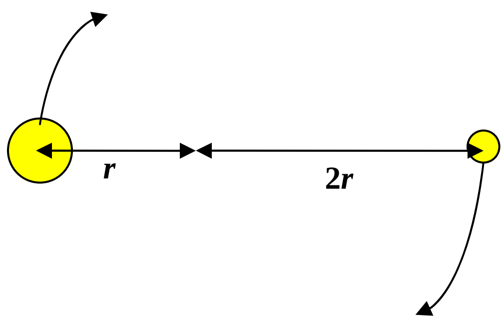


For this case, the equation $F = Ma$ must clearly be adjusted from the standard form above to:

$$\frac{GM^2}{(2r)^2} = \frac{Mv^2}{r} = Mr\omega^2$$

Problems with more than one rotating body turn out to handle more easily if the acceleration is written $r\omega^2$ rather than v^2/r. That's because ω is the same for the two stars, v isn't, except in the special case of equal mass.

Consider now a binary system in which one star has mass M, the other $2M$, but stay with the simple case of circular orbits. This time both stars go in circles around the common center of mass, and of course both move at the same angular velocity ω, so their angular accelerations are $2r\omega^2$, $r\omega^2$ respectively, the accelerations are in inverse proportion to the masses, as they must be since both experience the same magnitude force, their mutual gravitational attraction.



The Earth-Moon System: Tidal Forces

The Earth's mass is about eighty times the Moon's mass. This means that the Earth and the Moon both circle the system center of mass, a point about one-eightieth of the way from the center of the Earth to the center of the Moon—about 3,000 miles from Earth's center, so still inside the Earth.

To compute the Moon's orbital period, if we need to be precise, we should adjust the equation previously used to

$$mr_c\omega^2 = GMm/r^2$$

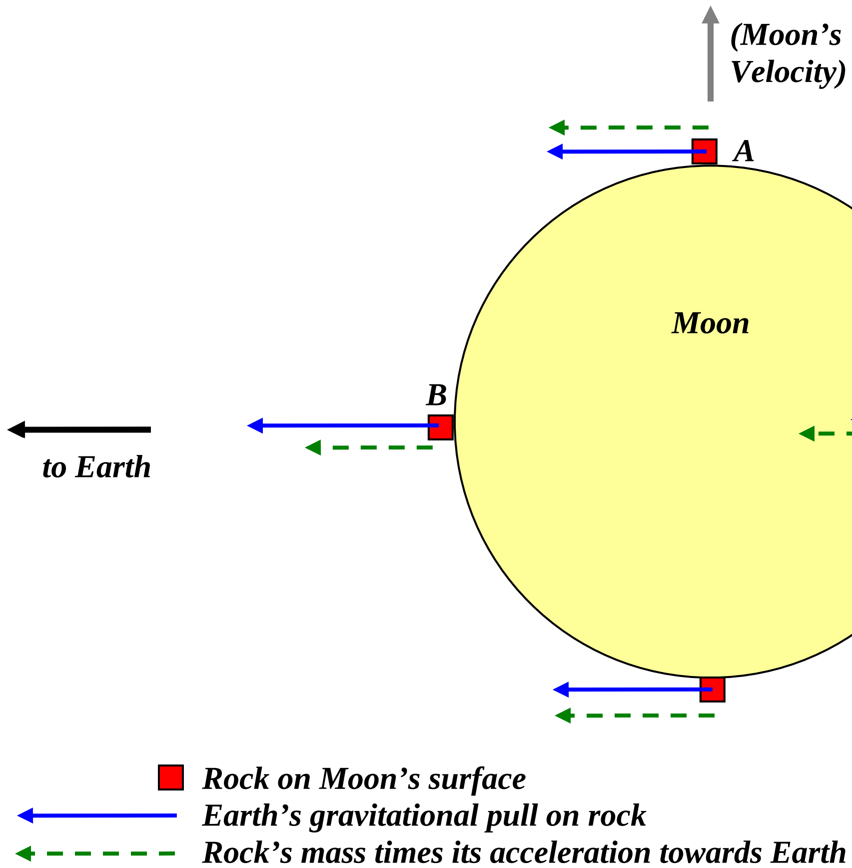
where r is the Earth-Moon distance, and r_c is the Moon's distance from the system center of mass. Putting $r_c = r$ gives close to 1% accuracy, usually adequate for our purposes here, but clearly not for precision astronomy.

Another important point is that to find the gravitational force on the Moon, we take it to be the same as if all the Moon's mass were concentrated at a point in the center. Assuming the Moon is spherically symmetrical, this is OK. We've established that the force on a mass outside the Moon is the same as if all the Moon's mass were at the center, so, since the gravitational forces are equal and opposite, the force on the Moon from the mass is the same as if all the Moon's mass were at the center. Therefore, the gravitational force of the Earth on the Moon, which can be thought of as the sum of all the one kilogram masses making up the Earth, must be the same as if all the Moon's mass were at the center.

Weighing Rocks on the Moon

To understand how the Earth's gravitational pull, and the Moon's orbital motion around the Earth, affect the apparent value of gravity on the surface of the Moon, we imagine having a set of identical rocks which we weigh with identical spring scales at different points of the Moon, as shown in the diagram below. Now, a spring scale is just a spring which compresses when a weight is placed on it, the amount of compression of the spring is linearly proportional to the weight added (within the design range of the scale) and as the spring compresses it turns a pointer hand around a dial. The dial then records the weight. Well, actually, to be precise, the dial records the force the spring is exerting on the rock: the normal force N, that is, the same force the rock would experience from the ground if it were just resting on the ground.

So, the forces on the rocks A, B, C shown are their weights W, all directed towards the center of the Moon, and all equal in magnitude, the force N from the compressed springs they're resting on (not shown), and the Earth's gravitational pull, the blue arrows in the diagram, *decreasing* with increasing distance from Earth. Since the rocks are going round the Earth with the Moon in its orbit, their accelerations towards the Earth are $r_c\omega^2$, this acceleration *increases* with distance from the Earth, since ω is the same for all of them.



Remember that the Earth's total gravitational pull on the Moon is the same as if all the Moon's mass were concentrated at the Moon's central point. If we assume rock *A* in the diagram is exactly the same distance from Earth as the center of the Moon is, it will feel the same gravitational pull towards Earth as a point mass at the center of the Moon, and therefore will accelerate towards Earth at the same rate: it will stay with the Moon, with no tendency to move towards or away from the Earth. Meanwhile, in the perpendicular direction, the spring balance it's resting on measures the force *N* with which the spring is supporting it, and this equals its weight *W*, meaning how strongly attracted it is by the Moon's gravity.

Now consider rock *B* on the left, closer to earth. It will feel a *stronger* gravitational pull from the Earth, than rock *A* does, yet its acceleration $r\omega^2$ is *less* than rock *A*'s acceleration—*r* is less, and ω is the same.

What about $\vec{F} = m\vec{a}$?

There must clearly be some force *opposing* the Earth's gravity, since *B*'s acceleration towards Earth is *less* than that given by the Earth's gravity acting alone. And there is: the Moon's gravity, the rock's weight *W*, pulls it the other way. But isn't the weight balanced by the spring force *N*? The answer is, it cannot be, since we always have $\vec{F} = m\vec{a}$. We are forced to conclude that the spring force *N*, the weight registered on the dial of the scale, is *less* than its true weight *W*.

The basic result is that on the part of the Moon closest to the Earth, things seem lighter: it's easier to lift something, gravity is effectively lessened by the Earth's pulling things "up".

Let's now look at rock *C*. Being further from the Earth, but going around with the Moon at the same angular velocity ω, its acceleration $r\omega^2$ is *greater* than rock *A*'s, but the Earth's gravitational pull is *weaker* at the greater distance.

Again, to satisfy $\vec{F} = m\vec{a}$, the Moon's gravitational pull *W* on the rock at *C*, must be out of balance with the force *N* from the spring. In fact, the net force from these two must point *left* (towards the Earth) to give the greater acceleration. Therefore, *the Moon's gravitational pull must be stronger than the normal force* from the surface. That means that a rock at *C* placed on a spring scale will register a smaller weight—the *same* effect as at *B*! Thing seem lighter at the *furthest* point from Earth as well!

This means that rocks at *B* and *C* will experience what amounts to *an apparently lower gravitational pull* to the Moon's center than a rock at *A*. Imagine now that the Moon were covered with an ocean. The effectively stronger gravity at places like *A* would pull the water down more than the weaker effective gravity at *B*, *C*. This is the origin of tides: the high tide is where "gravity" is weakest, on two opposite sides. Of course, there is no ocean on the Moon, but this same argument works for the effect of the Moon's gravity on the Earth: remember the Earth is also circling the Earth-Moon system center of mass.

Remarks on General Relativity

Einstein's Parable

In Einstein's little book *Relativity: the Special and the General Theory*, he introduces general relativity with a parable. He imagines going into deep space, far away from gravitational fields, where any body moving at steady speed in a straight line will continue in that state for a very long time. He imagines building a space station out there - in his words, "a spacious chest resembling a room with an observer inside who is equipped with apparatus." Einstein points out that there will be no gravity, the observer will tend to float around inside the room.

But now a rope is attached to a hook in the middle of the lid of this "chest" and an unspecified "being" pulls on the rope with a constant force. The chest and its contents, including the observer, accelerate "upwards" at a constant rate. How does all this look to the man in the room? He finds himself moving towards what is now the "floor" and needs to use his leg muscles to stand. If he releases anything, it accelerates towards the floor, and in fact all bodies accelerate at the same rate. If he were a normal human being, he would assume the room to be in a gravitational field, and might wonder why the room itself didn't fall. Just then he would discover the hook and rope, and conclude that the room was suspended by the rope.

Einstein asks: should we just smile at this misguided soul? His answer is no - the observer in the chest's point of view is just as valid as an outsider's. In other words, *being inside* the (from an outside perspective) *uniformly accelerating room is physically equivalent to being in a uniform gravitational field*. This is the basic postulate of general relativity. Special relativity said that all inertial frames were equivalent. General relativity extends this to accelerating frames, and states their equivalence to frames in which there is a gravitational field. This is called the *Equivalence Principle*.

The acceleration could also be used to cancel an existing gravitational field—for example, inside a freely falling elevator passengers are weightless, conditions are equivalent to those in the unaccelerated space station in outer space.

It is important to realize that this equivalence between a gravitational field and acceleration is only possible because the gravitational mass is exactly equal to the inertial mass. There is no way to cancel out electric fields, for example, by going to an accelerated frame, since many different charge to mass ratios are possible.

As physics has developed, the concept of fields has been very valuable in understanding how bodies interact with each other. We visualize the electric field lines coming out from a charge, and know that something is there in the space around the charge which exerts a force on another charge coming into the neighborhood. We can even compute the energy density stored in the electric field, locally proportional to the square of the electric field intensity. It is tempting to think that the gravitational field is quite similar—after all, it’s another inverse square field. Evidently, though, this is not the case. If by going to an accelerated frame the gravitational field can be made to vanish, at least locally, it cannot be that it stores energy in a simply defined local way like the electric field.

We should emphasize that going to an accelerating frame can only cancel a *constant* gravitational field, of course, so there is no accelerating frame in which the whole gravitational field of, say, a massive body is zero, since the field necessarily points in different directions in different regions of the space surrounding the body.

Some Consequences of the Equivalence Principle

Consider a freely falling elevator near the surface of the earth, and suppose a laser fixed in one wall of the elevator sends a pulse of light horizontally across to the corresponding point on the opposite wall of the elevator. Inside the elevator, where there are no fields present, the environment is that of an inertial frame, and the light will certainly be observed to proceed directly across the elevator. Imagine now that the elevator has windows, and an outsider at rest relative to the earth observes the light. As the light crosses the elevator, the elevator is of course accelerating downwards at g, so since the flash of light will hit the opposite elevator wall at precisely the height relative to the elevator at which it began, the outside observer will conclude that the flash of light also accelerates downwards at g. In fact, the light could have been emitted at the instant the elevator was released from rest, so we must conclude that light falls in an initially parabolic path in a constant gravitational field. Of course, the light is traveling very fast, so the curvature of the path is small! Nevertheless, *the Equivalence Principle forces us to the conclusion that the path of a light beam is bent by a gravitational field.*

The curvature of the path of light in a gravitational field was first detected in 1919, by observing stars very near to the sun during a solar eclipse. The deflection for stars observed very close to the sun was 1.7 seconds of arc, which meant measuring image positions on a photograph to an accuracy of hundredths of a millimeter, quite an achievement at the time.

One might conclude from the brief discussion above that a light beam in a gravitational field follows the same path a Newtonian particle would if it moved at the speed of light. This is true in the limit of small deviations from a straight line in a constant field, but is not true even for small deviations for a spatially varying field, such as the field near the sun the starlight travels through in the eclipse experiment mentioned above. We could try to construct the path by having the light pass through a series of freely falling (fireproof!) elevators, all falling towards the center of the sun, but then the elevators are accelerating relative to each other (since they are all falling along *radii*), and matching up the path of the light beam through the series is tricky. If it is done correctly (as Einstein did) it turns out that the angle the light beam is bent through is twice that predicted by a naïve Newtonian theory.

What happens if we shine the pulse of light vertically *down* inside a freely falling elevator, from a laser in the center of the ceiling to a point in the center of the floor? Let us suppose the flash of light leaves the ceiling at the instant the elevator is released into free fall. If the elevator has height h, it takes time h/c to reach the floor. This means the floor is moving downwards at speed gh/c when the light hits.

Question: Will an observer on the floor of the elevator see the light as Doppler shifted?

The answer has to be no, because inside the elevator, by the Equivalence Principle, conditions are identical to those in an inertial frame with no fields present. There is nothing to change the frequency of the light. This implies, however, that to an outside observer, stationary in the earth's gravitational field, the frequency of the light *will* change. This is because he will agree with the elevator observer on what was the initial frequency f of the light as it left the laser in the ceiling (the elevator was at rest relative to the earth at that moment) so if the elevator operator maintains the light had the same frequency f as it hit the elevator floor, which is moving at gh/c relative to the earth at that instant, the earth observer will say the light has frequency $f(1 + v/c) = f(1+gh/c^2)$, using the Doppler formula for very low speeds.

We conclude from this that light shining downwards in a gravitational field is shifted to a higher frequency. Putting the laser in the elevator floor, it is clear that light shining upwards in a gravitational field is red-shifted to lower frequency. Einstein suggested that this prediction could be checked by looking at characteristic spectral lines of atoms near the surfaces of very dense stars, which should be red-shifted compared with the same atoms observed on earth, and this was confirmed. This has since been observed much more accurately. An amusing consequence, since the atomic oscillations which emit the radiation are after all just accurate clocks, is that *time passes at different rates at different altitudes.* The US atomic standard clock, kept at 5400 feet in Boulder, gains 5 microseconds per year over an identical clock almost at sea level in the Royal Observatory at Greenwich, England. Both clocks are accurate to one microsecond per year. This means you would age more slowly if you lived on the surface of a planet with a large gravitational field. Of course, it might not be very comfortable.

General Relativity and the Global Positioning System

Despite what you might suspect, the fact that time passes at different rates at different altitudes has significant practical consequences. An important *everyday* application of general relativity is the Global Positioning System. A GPS unit finds out where it is by detecting signals sent from orbiting satellites at precisely timed intervals. If all the satellites emit signals simultaneously, and the GPS unit detects signals from four different satellites, there will be three relative time delays between the signals it detects. The signals themselves are encoded to give the GPS unit the precise position of the satellite they came from at the time of transmission. With this information, the GPS unit can use the speed of light to translate the detected time delays into distances, and therefore compute its own position on earth by triangulation.

But this has to be done very precisely! Bearing in mind that the speed of light is about one foot per nanosecond, an error of 100 nanoseconds or so could, for example, put an airplane off the runway in a blind landing. This means the clocks in the satellites timing when the signals are sent out must certainly be accurate to 100 nanoseconds a day. That is one part in 10^{12}. It is easy to check that both the special relativistic time dilation correction from the speed of the satellite, and the general relativistic gravitational potential correction are much greater than that, so the clocks in the satellites must be corrected appropriately. (The satellites go around the earth once every twelve hours, which puts them at a distance of about four earth radii. The calculations of time dilation from the speed of the satellite, and the clock rate change from the gravitational potential, are left as exercises for the student.) For more details, see the lecture by Neil Ashby here.

In fact, Ashby reports that when the first Cesium clock was put in orbit in 1977, those involved were sufficiently skeptical of general relativity that the clock was not corrected for the gravitational redshift effect. But—just in case Einstein turned out to be right—the satellite was equipped with a synthesizer that could be switched on if necessary to add the appropriate relativistic corrections. After letting the clock run for three weeks with the synthesizer turned off, it was found to differ from an identical clock at ground level by precisely the amount predicted by special plus general relativity, limited only by the accuracy of the clock. This simple experiment verified the predicted gravitational redshift to about one percent accuracy! The synthesizer was turned on and left on.

Gravity Fact Sheet

Gravitational acceleration g = **9.81 m/sec^2**.

Newton's Universal Gravitational Constant G **= 6.67 x 10^{-11} N.m^2/kg^2**.

The **Sun** has a **mass** of 1.99 x 10^{30} kg, and a **radius** of 6.96 x 10^8 m.

The **Moon** has a **mass** of 7.35 x 10^{22} kg, a **radius** of 1738 km., and an average **orbital radius** of 3.84 x 10^5 km., and **orbital period** of 27.3 days. The orbital **eccentricity** is 0.055, the orbit is **inclined** at 5.15 degrees to the Earth's orbit around the Sun.

Planetary Data

Planet	Radius (km)	Orbital Semimajor Axis(10^6km)	Mass (kg)	Orbital Period	Orbital Eccentricity	Inclination to Earth's Orbit
Mercury	2440	57.9	3.30 x 10^{23}	88 days	0.206	7.00
Venus	6050	108	4.87 x 10^{24}	225 days	0.00677	3.39
Earth	6380	150	5.97 x 10^{24}	1 yr	0.0167	0
Mars	3400	228	6.42 x 10^{23}	1.88 yr	0.0934	1.85
Jupiter	71,500	778	1.90 x 10^{27}	11.9	0.0484	1.31
Saturn	60,300	1430	5.69 x 10^{26}	29.4	0.0542	2.48
Uranus	25,600	2870	8.69 x 10^{25}	83.8	0.0472	0.770
Neptune	24,800	4500	1.02 x 10^{26}	164	0.00859	0.770
Pluto	1150	5920	1.31 x 10^{22}	248	0.249	17.1

From the AIP Physics Desk Reference, Third Edition, which gives original sources, and many more details. (Except Pluto mass: that's from http://arxiv.org/abs/astro-ph/0512491)

Astronomical Distance Units

1 AU = 1.5 x 10^{11}m. (AU = Astronomical Unit = Earth-Sun distance.)
1 light year = 9.46 x 10^{15} m.
1 parsec = 3.09 x 10^{16} m. (= 3.27 ly.) (The distance to a star that has apparent parallax movement caused by the Earth's orbital motion of one second of arc amplitude.)

Angle Measurement

1 radian $= 57.3°$, $1° = 60'$ (minutes), $1' = 60''$ (seconds).

Comparing Other Planets with the Earth

Planet	Radius compared with Earth's	Mass compared with Earth's	Orbital Period compared with Earth's	*g* at surface compared with Earth's
Mercury	0.382	0.0553	0.241	0.378
Venus	0.949	0.815	0.615	0.894
Earth	1	1	1	1
Mars	0.533	0.107	1.88	0.379
Jupiter	11.2	318	11.9	2.54
Saturn	9.41	95.2	29.4	1.07
Uranus	4.0	14.5	83.8	0.8
Neptune	3.9	17.2	164	1.2
Pluto	0.19	0.0021	248	0.059

And …

	Radius compared with Earth's	Mass compared with Earth's	*g* at surface compared with Earth's
Sun	109	3.33×10^5	$28g_{earth}$
Moon	0.272	0.0123	$0.166g_{earth}$

Neutron Stars, etc.

Neutron stars are formed when stars run out of nuclear fuel and collapse (see Wikipedia.) They have masses 1.35 to 2.1 solar masses, radii between 20 and 10 km (heavier ones are smaller!). Gravity *g* at the surface is 2×10^{11} to 3×10^{12} g_{earth}. Stars with less mass form white dwarfs, about the size of the Earth, but the mass of the Sun. Collapsing stars with masses above about 3 solar masses form black holes.

Energies

Sun's luminosity: 3.83×10^{26} J/sec.
1 megaton = 4.18×10^{15} J.

Physics 152: Gravity Homework (and some Exam) Problems

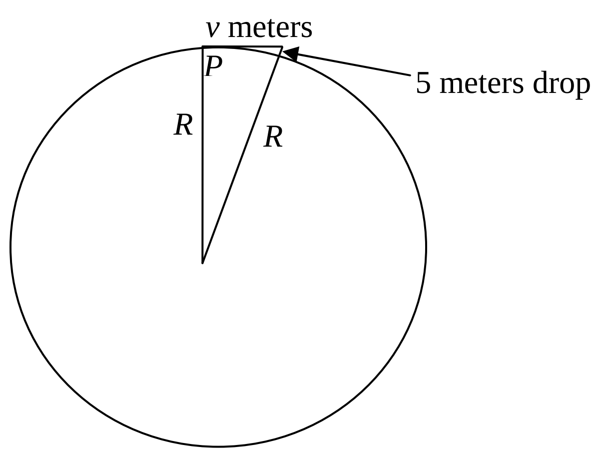
Michael Fowler

1. *Warm-up exercise: deriving acceleration in circular motion from Pythagoras' theorem.*

Imagine a cannon on a high mountain shoots a cannonball horizontally above the atmosphere at the right speed for it to go in a circular orbit. In one second, the ball will fall 5 meters below a horizontal line, at the same time traveling *v* meters horizontally, as in the diagram below (where the distances traveled are grossly exaggerated to make clear what's going on).

Apply Pythagoras' theorem to the right-angled triangle to establish that the appropriate speed for a circular orbit just above the earth's atmosphere is given by $v^2/r_E = g$.

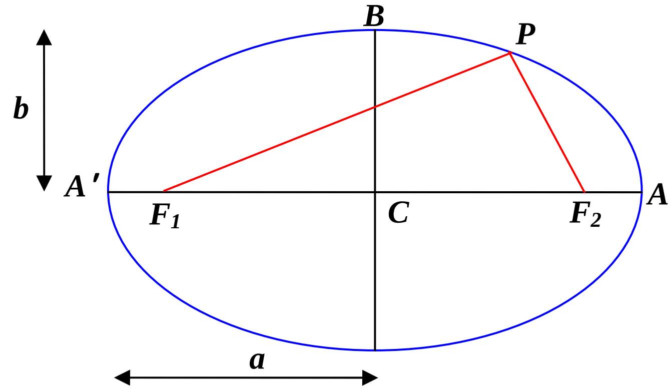
(Use the approximation that the distance traveled in one second is tiny compared to the radius of the earth.)



2. *Properties of the Ellipse.*

Take a point P' on the ellipse very close to P, and draw lines from the new point to the two foci.

Use the fact that the "rope" is the same length for P, P' to prove that a light ray from one focus to P will be reflected to the other focus.



3. Kepler's Third Law states that T^2/R^3 has the same numerical value for all the sun's planets.

For circular orbits, how are R, T related if the gravitational force is proportional to $1/R$? to $1/R^3$? To R? What can we conclude from Kepler's Third Law about the gravitational force?

4. Television signals are relayed by synchronous satellites, placed in orbits such that they hover above the same spot on Earth. Use Kepler's Laws and data about the Moon's orbit to find how far above the Earth's surface the synchronous satellites are. Could one be placed directly above Charlottesville? If you say no, explain your reasoning.

5. An evil genius puts a spherical rock (made of ordinary stone) in the earth's orbit, but moving around the sun *the other way*. It collides with the earth, landing in the desert. It is estimated that the crater is about the same as would have been caused by a one-megaton hydrogen bomb. How big was the rock?

6. Halley's comet follows an elliptical orbit, its closest approach to the Sun is observed to be 0.587AU. Given that the orbital period is 76 years, what is its furthest distance from the Sun? What is the ellipticity of this orbit?

7. *Halley's Comet simplified*.

(a) A comet having a period of 64 years has closest approach to the Sun 0.5 AU. Use Kepler's Third Law, and comparison with the Earth, to figure out its farthest distance from the Sun.

(b) What is the ratio of its kinetic energy when nearest to the Sun to its kinetic energy at the farthest point?

(c) How does its kinetic energy at the closest approach to the Sun compare with that of an equal mass in a circular orbit around the Sun at that distance? (An approximate answer will do.)

8. Galileo discovered four satellites of Jupiter:

Satellite	Orbital Radius in 10^6 km	Orbital Period in Days
Io	0.422	1.77
Europa	0.671	3.55
Ganymede	1.070	7.16
Callisto	1.880	16.7

The orbits are all very close to circular.

(Data from http://www.ifa.hawaii.edu/~sheppard/satellites/jupsatdata.html , where 59 other satellites of Jupiter are listed!)

Check that Kepler's Third Law is satisfied in this system, and use these data to find the mass of Jupiter.

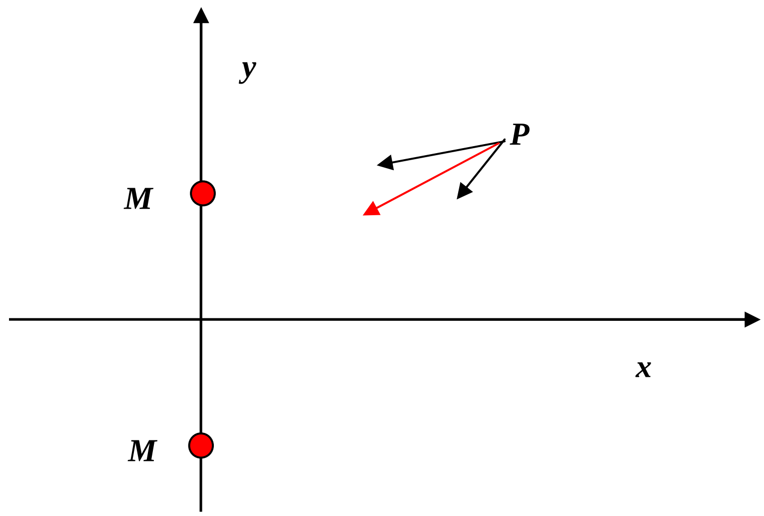
9. The galaxy NGC 4258 contains a disk of matter, like a huge version of Saturn's rings. The disk is not rigid, but is made up of rocks, etc., all going in approximately circular orbits. The disk has inner radius 0.14 pc (parsec), outer radius 0.28 pc. The inmost part is orbiting with a period of 800 yrs, the outer edge with a period of 2200 yrs.

(a) Show that these data indicate the disk is in a gravitational field dominated by a central massive object (rather than, say, the field of the disk itself).

(b) Find the approximate mass of the central object. The densest known star cluster is about 10^5 solar masses/pc^3. Could the central object be a star cluster? If not, what?

10. *Plotting the Gravitational Field.*

The diagram below shows how to find the gravitational force at a particular point from a system of two masses.



(a) Draw the field vector at several other points, then construct a picture of the field by drawing field lines: continuous lines which at each point are in the direction of the field at that point. (The same as "lines of force" in magnetism.)

(b) Draw the field line diagram for two *unequal* masses, such as the Earth and the Moon. In particular, make clear how the field behaves along the direct line from the Earth to the Moon.

11. (a) Give a brief explanation, with a diagram, of why the gravitational field inside a uniform spherical shell of matter is zero.

(b) Suppose a very deep tunnel is drilled vertically down. What is the gravitational force felt by a mass of 1 kg in the tunnel at a distance r from the center of the earth, given that it is 10 Newtons at distance R_E = 6400 km., that is, at the earth's surface? (Assume the Earth's density is uniform.)

** The rest of this question requires knowledge of Simple Harmonic Motion.*

(c) Now suppose in a massive engineering project the tunnel is drilled in a straight line through the center of the earth and reemerges near Australia. The air is pumped out of the tunnel, leaving a vacuum. A 1 kg package is dropped from rest at one end. How long does it take to reach the other end?

(d) Suppose there is an asteroid of 64 km radius, made of material with the same density as the earth. If an exactly similar tunnel is drilled through this asteroid, how long would it take a package to "fall" from one end to the other?

12. In the year 3000, a group of bad guys fond of living in caves have excavated a huge spherical cave inside the Moon. (But it's not centered at the center of the Moon!) Assuming the Moon is a sphere of rock of uniform density, prove that the gravitational field inside the cave is the same everywhere. (Hint: figure out the field for Moon with no cave, then think of the cave as a uniform sphere of *negative* mass density, and add the two contributions.)

13. Imagine a tunnel bored straight through the Earth emerging at the opposite side of the globe. The gravitational force on a mass *m* in the tunnel is $F = mgr / r_E$.

(a) Find an expression for the gravitational *potential* in the tunnel. Take it to be zero at the center of the Earth.

(b) Now sketch a graph of the potential as a function of distance from the Earth's center, beginning at the center but continuing beyond the Earth's radius to a point far away. This curve must be continuous.

(c) Conventionally, the potential energy is defined by requiring it to be zero at infinity. How would you adjust your answer to give this result?

14. Draw a plot of the gravitational *potential* along a straight line from the surface of the Earth to the surface of the Moon. What is the minimum speed of a rocket fired directly from the Earth to the Moon to reach it? What speed will it be moving on reaching the Moon's surface? (Ignore the Earth's rotation and the Moon's orbital speed—just consider two fixed masses.)

15. *For this question, take the mass of Mars to be 0.1 Earth masses, and the radius of Mars to be 0.5 Earth radii.*

(a) Given that g = 10 m/sec^2 at the Earth's surface, what is the acceleration due to gravity at the surface of Mars? (Show your working.)

(b) A satellite in low Earth orbit travels at 8 km/sec. Use the value of g on Mars you found in part (a) to work out how fast a satellite in a low Mars orbit will travel.

(c) Calculate the escape velocity from Mars.

(d) A synchronous satellite is in a circular orbit (around the Earth) with radius 42,000 km. It happens that the length of a Martian day is close to 24 hours. What would be the orbit radius of a synchronous satellite circling Mars?

16. Phobos, a satellite of Mars, has a radius of 11 km and a mass of 10^{16} kg. It's a bit lumpy, but let's assume it's spherical to get a doable problem.

(a) What is g on Phobos?

(b) If you can jump to a height of one meter on earth, how high could you jump on Phobos? (Think carefully about this.)

(c) Could an astronaut on a bicycle reach orbital speed on Phobos? (Guesses don't count, I need to see a derivation). What about reaching escape velocity?

17. Uranus has a radius four times Earth's radius, but gravity at the surface is only $0.8g_{earth}$. Escape velocity from Earth is 11.2 km/sec. Using these facts, and nothing else, find the escape velocity from Uranus.

18. (a) Find the orbital speed of a spaceship in low orbit around the Moon, just skimming the mountain tops.

(b) Suppose the pilot suddenly increases the speed by a factor of $\sqrt{2}$, but during the brief acceleration keeps the spaceship pointing the same way, that is, horizontally. Describe the path the spaceship will take after the engine cuts out—does this curve have a name?.

19. In deep space, an astronaut is marooned ten meters from his four-ton spacecraft. If he is exactly at rest relative to the craft, and there are no other gravitational fields close by, *estimate* how long it will be before he's back on board. How fast will he be moving when he hits the craft (which is 5 meters in diameter)?

20. The escape speed from the moon is 2.38 km/sec. Suppose you had on the moon a cannon that could fire shells at 2.4 km/sec. Obviously, if you fired a shell vertically upwards, it would escape the moon's gravity. But what if you fired it almost horizontally, just elevated enough so it cleared the mountains? Describe its trajectory in this situation.

21. The escape velocity from Earth is 11.2 km./sec. What is the escape velocity from *the Solar System* starting in a high parking orbit several Earth radii from Earth. (Hint: what is the Earth's speed in orbit?) On the basis of this, estimate roughly how much more fuel energy is needed to reach the outer planets compared with going to the Moon. Is there a way around this problem?

22. Imagine a fictitious moon, which we'll call Moon1, a sphere with the same density as the earth, but with radius exactly one-quarter the earth's radius:

$$\rho_{\text{Moon1}} = \rho_{\text{earth}},\ \ R_{\text{Moon1}} = 0.25 R_{\text{earth}}.$$

(a) Taking the acceleration due to gravity g_{earth} to be 10 m.sec^{-2} at the earth's surface, what is the acceleration g_{Moon1} due to Moon1's gravity at Moon1's surface?

(b) If the escape velocity from the earth's surface is 11 km.sec^{-1}, what is the escape velocity from Moon1's surface?

(c) If it takes 90 minutes for a satellite in low earth orbit (orbit radius approximately equal to earth radius) to go around once, how long will it take a satellite in a low Moon1 orbit (skimming the surface of the airless moon) to go around once?

(d) The real Moon has a radius close to that of Moon1 above (our Moon's radius is 10% bigger than one-quarter the earth's, we'll neglect that difference here). However, the real Moon has a density only 60% that of the earth. In this part, use the real Moon's density (but Moon1's radius) to recalculate the answers to (a), (b) and (c) above.

23. Saturn's satellite Titan has an orbit of radius 1.22 x 10^6 km., and a period of 15.9 days. Use this information to find the mass of Saturn, then use its radius of 60,300 km to deduce

(a) Saturn's average density

(b) the value of *g* at the surface of Saturn

(c) the escape velocity from Saturn.

24. The furthest planet, Pluto, has a radius 20% of the Earth's radius, and a mass only 0.2% that of the Earth. (Both figures are within about 5%.)

(a) Suppose an astronaut, in full insulated gear, can jump 0.5 m high on Earth. How high can she jump on Pluto? (You don't need to know *G* to answer this!)

(b) Assuming "air" resistance is negligible, what speed would a (rocket driven) car racing over a flat plane (a frozen sea) on Pluto need to be traveling to attain escape velocity? (Escape velocity from *Earth* is 11.2 km per sec: use this fact.)

(c) Would it in fact have left the ground before reaching that speed? Explain your answer.

25. The escape velocity from a certain planet is 10 km per sec. The planet has a moon having radius one-quarter that of the planet, and density one-half that of the planet. What is the escape velocity from the planet's moon?

26. In an imaginary universe, the gravitational force decreases with distance as $1/R$ instead of $1/R^2$. Suppose in that universe there is a planet the same size as Earth and also having the same value of *g* near the surface. Would the period of a satellite in low circular orbit (just above an atmosphere of negligible depth) be the same? Would the escape velocity be the same?

***27.** Somewhere on the line from the Earth to the Sun there is a point, called a Lagrange point, such that a satellite placed there will orbit around the Sun in sync with the Earth. In fact, there's already a satellite there, it monitors the Sun continuously. Come up with some estimate of how far from Earth this Lagrange point is (the Web might be helpful).

28. On a Moon mission, a spaceship is fired from Earth with just enough speed to reach the Moon, but aimed so that it just misses the Moon, and loops behind it, closest approach being near the point on the Moon's surface furthest from Earth. At that point, a small distance above the Moon's surface, the ship fires a rocket to put it into low circular orbit around the Moon. What is (approximately) the change in speed needed for this maneuver?

Elliptic Paths to Planets and Asteroids

29. The asteroid Gaspra is *twice as far from the sun* as we are. Assume it is in a circular orbit, and you are planning an expedition there.

The most economical trajectory is along an elliptical orbit, whose closest approach to the sun, call it r_1, is at the earth's orbit, and furthest distance from the sun, r_2, is at Gaspra's orbit. Suppose that after leaving the atmosphere, the spaceship is rapidly speeded up to v_1, then the engines cut out, and it follows the assigned elliptic path, arriving at Gaspra's orbit with speed v_2. (Neglect the earth's gravitational pull on the spaceship.)

(a) What quantities are conserved on the elliptic orbit?

(b) Find *two* equations for v_1, v_2 in terms of r_1, r_2 and GM, where M is the mass of the sun.

(c) Solve the two equations to find v_1.

(d) Find the speed of the earth in orbit in terms of r_1 and GM.

(e) Given that the earth's speed in orbit is 30 km per sec, how much does the spaceship need to be speeded up *relative to the earth* to get to Gaspra along this ellipse?

(f) Show on a diagram the earth in orbit, and the direction in which the spaceship needs to be moving just after leaving the earth to reach Gaspra. Approximately, what path would the spaceship take if fired in the opposite direction?

30. Suppose we are sending a space probe of mass m from Earth to Jupiter by the most economical elliptical route. Take the radius of Jupiter's orbit around the sun to be 5 AU.

(a) What is the total energy of the probe in the elliptical orbit?

(b) Assume it is fired from a parking orbit circling the earth far above, so the earth's own gravity has a negligible effect. Given that the earth moves in orbit at 30 km/sec, what is the speed of the probe relative to earth as it enters the elliptical orbit?

(c) What is its speed when it reaches Jupiter's orbit?

31. We plan to send a probe to an asteroid which has a circular orbit of radius three times that of the earth's orbit (assumed also circular).

(a) Sketch the most efficient path, showing on your diagram the earth's orbit and the asteroid's.

(b) If the earth travels in its orbit at 30 km per sec, at what speed relative to the earth must the probe be moving after it has cleared essentially all the earth's gravitational field?

32. Suppose a satellite is in low earth orbit, that is, in a circular orbit at a height of 200 km., so the radius of the circle is 6600 km., say. We want to raise it to a circular orbit of *twice* that radius (so it will now be going in a circle at a height of 6800 km *above the earth's surface*.)

The technique is to give it two quick boosts: boost1 puts it into an elliptical orbit, where its furthest point from the earth's center is exactly twice its distance of closest approach, boost2, delivered at the topmost point of the orbit, transfers it to a circular orbit at that radius.

Use conservation of angular momentum and energy in the elliptical orbit to answer these two questions:

(a) By what percentage did boost1 increase its speed?

(b) By what percentage did boost2 increase its speed?

(b) Give a qualitative explanation of how you would fire a rocket to get *back* to Earth from a parking orbit near Mars (so you neglect Mars' own gravity).

33. *A "Binary" System.*

A very recently discovered "earthlike" planet—we'll call it *P*—orbits the red dwarf star Gleise 581, which is 20 light years from us.
P's sun (Gleise 581) *has a mass one-third the mass of our sun*.

The planet *P*'s presence was established by detecting a wobble in the motion of Gleise 581 with a period of 13 days. (The wobble being caused by the orbiting planet's gravity: think binary system.)

(3) (a) How far is *P* from its sun?
Do this as follows: for any solar type system with circular planetary orbits $T^2/R^3 = 4\pi^2/GM$, M being the mass of that sun.

Use as units earth years and A.U. (distance of Earth from Sun), so for *our* solar system in these units $T^2/R^3 = 1$.

What is T^2/R^3 *in the same units* (earth years and earth A.U.'s) in the Gleise 581 system?

The wobble means *P* orbits its sun once in 13 days. Write that in earth years, and deduce how far *P* is from its sun, in A.U.

(b) Given that 1 A.U = 1.5x 10^8 km, how fast is *P* moving in its (assumed circular) orbit?

(c) From detecting the Doppler shift, it is found that the maximum speed of the sun Gleise 581 in its 13-day wobble is 3 m/sec. From this, thinking of the planet *P* and the sun Gleise 581 as a "binary star" system, what is the ratio of the planet *P* mass to the sun (Gleise) mass?

(d) Our sun's mass is about 300,000 earth masses. How does the "earthlike" planet *P*'s mass compare with the earth's mass? (Recall Gleise has a mass one-third of our sun's mass.)

General Relativity

34. (a) State the Equivalence Principle.

(b) Explain how shining light across an elevator can lead to the conclusion that light is deflected by gravity. (Include a diagram.)

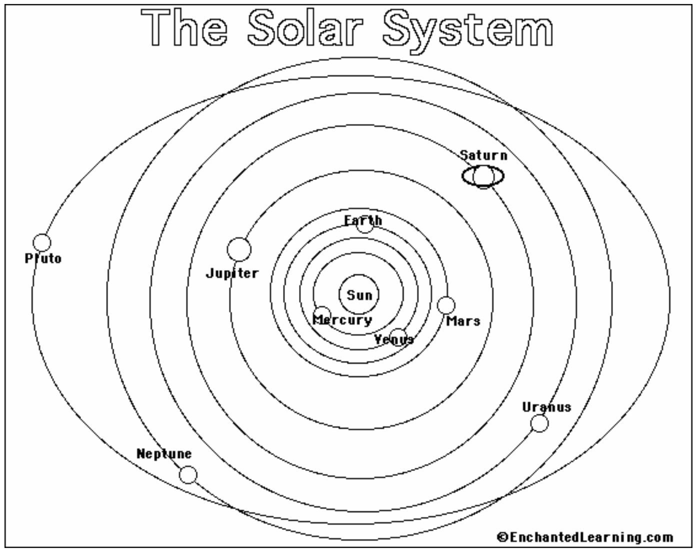
(c) Given that light is deflected of order 1 second of arc on passing by the sun, and that the order of magnitude is correctly given by a simple classical approximation, how much would you estimate light to be deflected (order of magnitude) passing the surface of a neutron star, having twice the mass of the Sun and a radius of 10 km (the sun's radius being 700,000 km)? State what approximations you're making.

35. The first experimental test of General Relativity was an observation of the deflection of starlight by the Sun's gravitational field (observed during a Solar eclipse). Classically, regarding light as tiny particles, the deflection can be estimated within 20% or so by approximating the Sun's gravitational effect as equal to gravity at the Sun's surface acting for a period of time equal to that needed for the particles to travel a distance equal to the Sun's diameter. Calculate what angular deflection that would give, in seconds of arc. General Relativity predicts that the actual deflection should be twice the classical value—and that was observed.

36. The GPS satellites are at an altitude of about 20,000 km. Find their speed, and figure out the necessary correction factors for their clocks from both Special and General Relativistic effects. Are these corrections important for the functioning of the system, or can they be neglected in practice?

Miscellaneous

37. At http://www.enchantedlearning.com/subjects/astronomy/activities/coloring/Solarsystem.shtml you will find the following image:



What's wrong with the orbit of Pluto as shown here?

38. Mercury can be observed as a small black dot crossing the face of the Sun, this occurs about every ten years on average. Since Mercury goes around in 88 days, why is this so rare? Also, it only ever happens in May or November. How would you explain this pattern?

39. Some future explorer decides to fly by a neutron star, following a free-fall trajectory in which the spaceship loops behind the neutron star and comes back—so within the spaceship, the astronaut will be "weightless". However, if *g* varies significantly between the head and foot of the astronaut, this could have disastrous consequences.

(a) Estimate at what rate of variation of *g* the astronaut is unsafe.

(b) Assume the neutron star has a mass of two solar masses, and a radius of 10 km. How close to the surface is it safe for the ship to approach?

40. The asteroid Icarus was only four million miles from earth on a recent pass. If a collision took place, and Icarus fell to earth, give a *ballpark* estimate of the energy released in the inelastic collision. Compare it with a one megaton hydrogen bomb.

Flashlet and Applet Exercises

41. (a) Activate the Mars trip flashlet. The initial launch speed you enter is from a high parking orbit (say at ten Earth radii) so that the Earth's own gravitational field has negligible effect. Find the minimum speed needed to reach Mars, sketch a picture of this most economical orbit, and estimate how long the trip would take.

(b) Give a qualitative explanation of how you would fire a rocket to get *back* to Earth from a parking orbit near Mars (so you neglect Mars' own gravity).

42. (a) From the Fact Sheet, find the speed of Jupiter in orbit.

(b) Open the Jupiter slingshot flashlet. Note that you can adjust both the initial speed of the rocket approaching Jupiter, and how closely you begin to Jupiter's orbit. Imagine your rocket barely makes it out to Jupiter's orbit, so has no speed left—but you've perfectly timed it to derive maximum benefit from the slingshot effect. Could Jupiter give it enough of a boost to get it thrown out of the solar system? Justify your answer: what does the rocket's orbit look like as seen by someone on Jupiter?

43. Open the Newtonian Mountain applet. The height of the mountain (Newton's own drawing) is about 10% of the Earth's diameter. This happens to be approximately the maximum height reached by an ICBM on a trajectory going half way around the world, so the cannonball path is the second half of an ICBM trajectory.

(a) Experiment with the applet to find the speed at the top of an ICBM trajectory compared with speed in a circular orbit at the same height.

(b) If the ICBM is launched with an engine that cuts out as it leaves the atmosphere, what is the approximate speed as the engine cuts out? (Answer in km/sec – I apologize for the applet being in mph. Take the radius of the Earth to be 6400 km., and neglect the thickness of the atmosphere.)

(c) When the engine cuts out, what is the angle between the trajectory and the horizontal? (*Hint*: use conservation of angular momentum.)