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Viscosity

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Introduction: Friction at the Molecular Level

Viscosity is, essentially, fluid friction. Like friction between moving solids, viscosity transforms kinetic energy of (macroscopic) motion into heat energy. Heat is energy of random motion at the molecular level, so to have any understanding of how this energy transfer takes place, it is essential to have some picture, however crude, of solids and/or liquids sliding past each other as seen on the molecular scale.

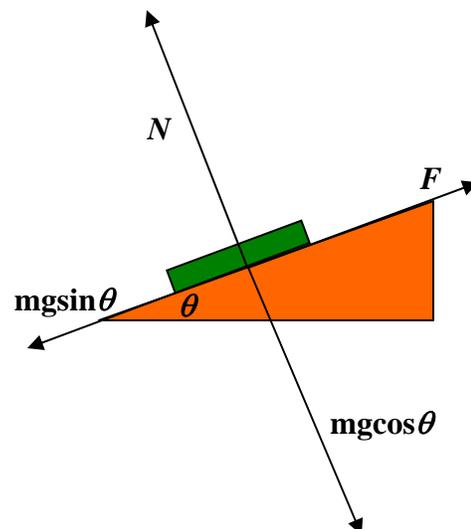
To begin with, we'll review the molecular picture of friction between *solid* surfaces, and the significance of the coefficient of friction μ in the familiar equation $F = \mu N$. Going on to fluids, we'll give the definition of the coefficient of viscosity for liquids and gases, give some values for different fluids and temperatures, and demonstrate how the microscopic picture can give at least a qualitative understanding of how these values vary: for example, on raising the temperature, the viscosity of liquids *decreases*, that of gases *increases*. Also, the viscosity of a gas doesn't depend in its density! These mysteries can only be unraveled at the molecular level, but there the explanations turn out to be quite simple.

As will become clear later, the coefficient of viscosity η can be viewed in two rather different (but of course consistent) ways: it is a measure of how much heat is generated when faster fluid is flowing by slower fluid, but it is also a measure of the *rate of transfer of momentum* from the faster stream to the slower stream. Looked at in this second way, it is analogous to thermal conductivity, which is a measure of the rate of transfer of heat from a warm place to a cooler place.

Quick Review of Friction Between Solids

First, *static* friction: suppose a book is lying on your desk, and you tilt the desk. At a certain angle of tilt, the book begins to slide. Before that, it's held in place by "static friction". What does that mean on a molecular level? There must be some sort of attractive force between the book and the desk to hold the book from sliding.

Let's look at all the forces on the book: gravity is pulling it vertically down, and there is a "normal force" of the desk surface pushing the book in the direction normal to the desk surface. (This normal force is the springiness of the desktop, slightly compressed by the weight of the book.) When the desk is tilted, it's best to visualize the vertical gravitational force as made up of a component normal to the surface and one parallel to the surface (downhill). The gravitational



component perpendicular to the surface is exactly balanced by the normal force, and if the book is at rest, the “downhill” component of gravity is balanced by a frictional force parallel to the surface in the uphill direction. On a microscopic scale, this static frictional force is from fairly short range attractions between molecules on the desk and those of the book.

Question: but if that’s true, why does doubling the normal force double this frictional force? (Recall $F = \mu N$, where N is the normal force, F is the limiting frictional force just before the book begins to slide, and μ is the coefficient of friction. By the way, the first appearance of F being proportional to N is in the notebooks of Leonardo da Vinci.)

Answer : Solids are almost always *rough* on an atomic scale: when two solid surfaces are brought into contact, in fact only a tiny fraction of the common surface is really in contact at the atomic level. The stresses within that tiny area are large, the materials distort plastically and there is adhesion. The picture can be very complex, depending on the materials involved, but the bottom line is that there is only atom-atom interaction between the solids *over a small area*, and what happens in this small area determines the frictional force. If the normal force is doubled (by adding another book, say) the tiny area of contact between the bottom book and the desk will also double—the true area of atomic contact *increases linearly with the normal force*—that’s why friction is proportional to N . Within the area of “true contact” extra pressure makes little difference. (Incidentally, if two surfaces which really *are* flat at the atomic level are put together, there is bonding. This can be a real challenge in the optical telecommunications industry, where wavelength filters (called etalons) are manufactured by having extremely flat, highly parallel surfaces of transparent material separated by distances comparable to the wavelength of light. If they touch, the etalon is ruined.)

On tilting the desk more, the static frictional force turns out to have a limit—the book begins to slide. But there’s still some friction: experimentally, the book does not have the full acceleration the component of gravity parallel to the desktop should deliver. This must be because in the area of contact with the desk the two sets of atoms are constantly colliding, loose bonds are forming and breaking, some atoms or molecules fall away. This all causes a lot of atomic and molecular vibration at the surface. In other words, some of the gravitational potential energy the sliding book is losing is ending up as heat instead of adding to the book’s kinetic energy. This is the familiar dynamic friction you use to warm your hands by rubbing them together in wintertime. It’s often called *kinetic* friction. Like static friction, it’s proportional to the normal force: $F = \mu_k N$. The proportionality to the normal force is for the same reason as in the static case: the kinetic frictional drag force also comes from the tiny area of true atomic contact, and this *area* is proportional to the normal force.

A full account of the physics of friction (known as tribology) can be found, for example, in *Friction and Wear of Materials*, by Ernest Rabinowicz, second Edition, Wiley, 1995.

Liquid Friction

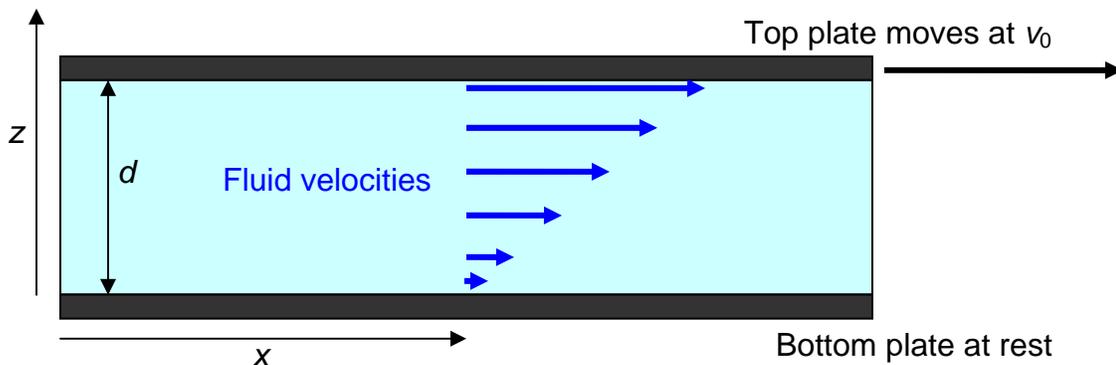
What happens if instead of two solid surfaces in contact, we have a solid in contact with a liquid? First, there's no such thing as static friction between a solid and a liquid. If a boat is at rest in still water, it will move in response to the slightest force. Obviously, a tiny force will give a tiny acceleration, but that's quite different from the book on the desk, where a considerable force gave no acceleration at all. But there *is* dynamic liquid friction—even though an axle turns a lot more easily if oil is supplied, there is still *some* resistance, the oil gets warmer as the axle turns, so work is being expended to produce heat, just as for a solid sliding across another solid.

One might think that for solid/liquid friction there would be some equation analogous to $F = \mu_k N$: perhaps the liquid frictional force is, like the solid, proportional to pressure? But experimentally this turns out to be false—there is little dependence on pressure over a very wide range. The reason is evidently that since the liquid can flow, there is good contact over the whole common area, even for low pressures, in contrast to the solid/solid case.

Newton's Analysis of Viscous Drag

Isaac Newton was the first to attempt a quantitative definition of a coefficient of viscosity. To make things as simple as possible, he attempted an experiment in which the fluid in question was sandwiched between two large parallel horizontal plates. The bottom plate was held fixed, the top plate moved at a steady speed v_0 , and the drag force from the fluid was measured for different values of v_0 , and different plate spacing. (Actually Newton's experiment didn't work too well, but as usual his theoretical reasoning was fine, and fully confirmed experimentally by Poiseuille in 1849 using liquid flow in tubes.)

Newton assumed (and it has been amply confirmed by experiment) that at least for low speeds the fluid settles into the flow pattern shown below. The fluid in close contact with the bottom plate stays at rest, the fluid touching the top plate gains the same speed v_0 as that plate, and in the space between the plates the speed of the fluid increases linearly with height, so that, for example, the fluid halfway between the plates is moving at $\frac{1}{2} v_0$:



Just as for kinetic friction between solids, to keep the top plate moving requires a steady force. Obviously, the force is proportional to the total amount of fluid being kept in

motion, that is, to the total area of the top plate in contact with the fluid. The significant parameter is the *horizontal force per unit area of plate*, F/A , say. This clearly has the same dimensions as pressure (and so can be measured in Pascals) although it is physically completely different, since in the present case the force is *parallel* to the area (or rather to a line within it), not perpendicular to it as pressure is.

(*Note for experts only*: Actually, viscous drag and pressure are not completely unrelated—as we shall discuss later, the viscous force may be interpreted as a rate of transfer of momentum into the fluid, momentum parallel to the surface that is, and pressure can *also* be interpreted as a rate of transfer of momentum, but now perpendicular to the surface, as the molecules bounce off. Physically, the big difference is of course that the pressure doesn't have to do any work to keep transferring momentum, the viscous force does.)

Newton conjectured that the necessary force F/A would be proportional to the velocity gradient in the vicinity of the top plate. In the simple geometry above, the velocity gradient is the same everywhere between the plates, v_0/d , so

$$F / A = \eta v_0 / d$$

defines the *coefficient of viscosity* η . The SI units of η are **Pascal.seconds**, or **Pa.s**. A convenient unit is the milliPascal.second, mPa.s. (It happens to be close to the viscosity of water at room temperature.) Confusingly, there is another set of units out there, the poise, named after Poiseuille—usually seen as the centipoise, which happens to equal the millipascal.second! And, there's *another* viscosity coefficient in common use: the kinetic viscosity, $\nu = \mu / \rho$, where ρ is the fluid density. This is the relevant parameter for fluids flowing downwards gravitationally. But we'll almost always stick with η .

Here are some values of η for common liquids:

Liquid	Viscosity in mPa.s
Water at 0°C	1.79
Water at 20°C	1.002
Water at 100°C	0.28
Glycerin at 0°C	12070
Glycerin at 20°C	1410
Glycerin at 30°C	612
Glycerin at 100°C	14.8
Mercury at 20°C	1.55
Mercury at 100°C	1.27
Motor Oil SAE 30	200
Motor Oil SAE 60	1000
Ketchup	50,000

Some of these are obviously ballpark – the others probably shouldn't be trusted to be better than 1% or so, glycerin maybe even 5-10% (see CRC Tables); these are quite difficult measurements, very sensitive to purity (glycerin is hygroscopic) and to small temperature variations.

To gain some insight into these very different viscosity coefficients, we'll try to analyze what's going on at the molecular level.

A Microscopic Picture of Viscosity in Laminar Flow

For Newton's picture of a fluid sandwiched between two parallel plates, the bottom one at rest and the top one moving at steady speed, the fluid can be pictured as made up of many layers, like a pile of printer paper, each sheet moving a little faster than the sheet below it in the pile, the top sheet of fluid moving with the plate, the bottom sheet at rest. This is called *laminar flow*: laminar just means sheet (as in laminate, when a sheet of something is glued to a sheet of something else). If the top plate is gradually speeded up, at some point laminar flow becomes unstable and turbulence begins. We'll assume here that we're well below that speed.

So where's the friction? It's not between the fluid and the plates (or at least very little of it is—the molecules right next to the plates mostly stay in place) it's *between the individual sheets*—throughout the fluid. Think of two neighboring sheets, the molecules of one bumping against their neighbors as they pass. As they crowd past each other, on average the molecules in the faster stream are slowed down, and those in the slower stream speeded up. Of course, *momentum* is always conserved, but the macroscopic kinetic energy of the sheets of fluid is partially lost—transformed into heat energy.

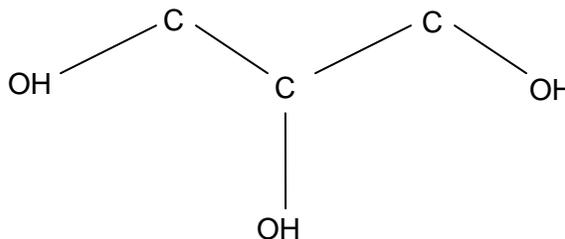
Exercise: Suppose a mass m of fluid moving at v_1 in the x -direction mixes with a mass m moving at v_2 in the x -direction. Momentum conservation tells us that the mixed mass $2m$ moves at $\frac{1}{2}(v_1 + v_2)$. Prove that the total kinetic energy has *decreased* if v_1, v_2 are unequal.

This is the fraction of the kinetic energy that has disappeared into heat.

This molecular picture of sheets of fluids moving past each other gives some insight into why viscosity decreases with temperature, and at such different rates for different fluids. As the molecules of the faster sheet jostle past those in the slower sheet, remember they are all jiggling about with thermal energy. The jiggling helps break them loose if they get jammed temporarily against each other, so as the temperature increases, the molecules jiggle more furiously, unjam more quickly, and the fluid moves more easily—the viscosity goes down.

This drop in viscosity with temperature is dramatic for glycerin. A glance at the molecule suggests that the zigzaggy shape might cause jamming, but the main cause of the stickiness is that the outlying H's in the OH groups readily form hydrogen bonds (see *Atkins' Molecules*, Cambridge).

For mercury, a fluid of round atoms, the drop in viscosity with temperature is



Glycerin Molecule: other carbon valence bonds have H atoms.

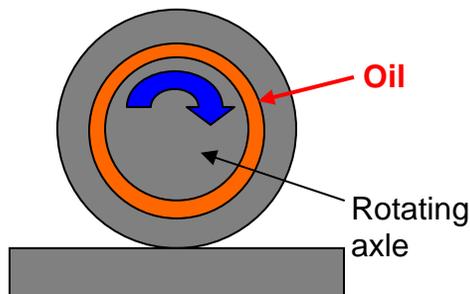
small. Mercury atoms don't jam much, they mainly just bounce off each other (but even that bouncing randomizes their direction, converting macroscopic kinetic energy to heat). Water molecules are in between glycerin and mercury in complexity. Looking at the table above, it is evident this simple picture makes at least qualitative sense of the data.

Another mechanism generating viscosity is the diffusion of faster molecules into the slower stream and vice versa. As discussed below, this is far the dominant factor in viscosity of gases, but is *much* less important in liquids, where the molecules are crowded together and constantly bumping against each other.

This temperature dependence of viscosity is a real problem in lubricating engines that must run well over a wide temperature range. If the oil gets too runny (that is, low viscosity) it will not keep the metal surfaces from grinding against each other; if it gets too thick, more energy will be needed to turn the axle. "Viscostatic" oils have been developed: the natural decrease of viscosity with temperature ("thinning") is counterbalanced by adding polymers, long chain molecules at high temperatures that curl up into balls at low temperatures.

Oiling a Wheel Axle

The simple linear velocity profile pictured above is actually a good model for ordinary lubrication. Imagine an axle of a few centimeters diameter, say about the size of a fist, rotating in a bearing, with a 1 mm gap filled with SAE 30 oil, having $\eta = 200$ mPa.s. (Note: mPa, millipascals, not Pascals! $1\text{Pa} = 1000\text{mPa}$.)



If the total cylindrical area is, say, 100 sq cm., and the speed is $1\text{ m}\cdot\text{s}^{-1}$, the force per unit area (sq. m.)

$$F / A = \eta v_0 / d = 200 \cdot 10^{-3} \cdot 1 / 10^{-3} = 200\text{ N} / \text{m}^2.$$

So for our 100 sq.cm bearing the force needed to overcome the viscous "friction" is 2N . At the speed of $1\text{ m}\cdot\text{sec}^{-1}$, this means work is being done at a rate of 2 joules per sec., or 2 watts, which is heating up the oil. (This heat must be conducted away, or the oil continuously changed by pumping, otherwise it will get too hot.)

*Viscosity: Kinetic Energy Loss and Momentum Transfer

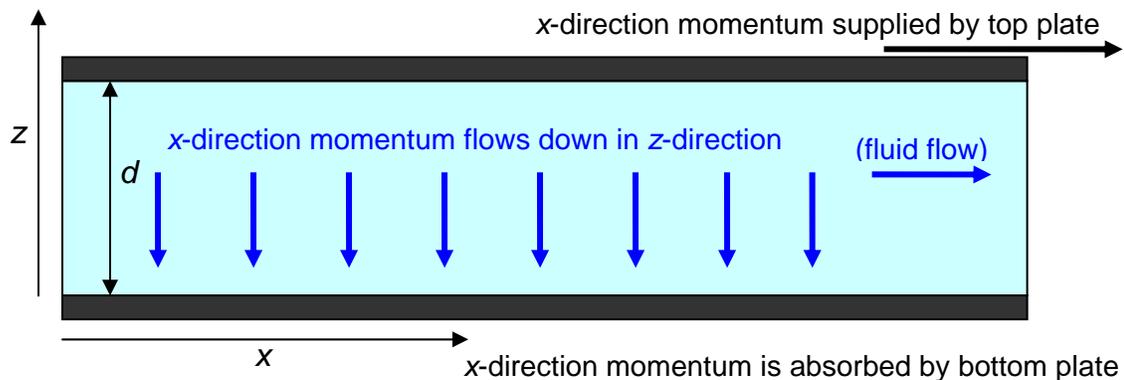
So far, we've viewed the viscosity coefficient η as a measure of friction, of the dissipation into heat of the energy supplied to the fluid by the moving top plate. But η is also the key to understanding what happens to the *momentum* the plate supplies to the fluid.

For the picture above of the steady fluid flow between two parallel plates, the bottom plate at rest and the top one moving, a steady force per unit area F/A in the x -direction applied to the top plate is needed to maintain the flow.

From Newton's law $F = dp/dt$, F/A is the rate at which momentum in the x -direction is being supplied (per unit area) to the fluid. Microscopically, molecules in the immediate vicinity of the plate either adhere to it or keep bouncing against it, picking up momentum to keep moving with the plate (these molecules also constantly lose momentum by bouncing off other molecules a little further away from the plate).

Question: But doesn't the total momentum of the fluid stay the same in steady flow? Where does the momentum fed in by the moving top plate go?

Answer: the x -direction momentum supplied at the top passes downwards from one layer to the next, ending up at the bottom plate (and everything it's attached to). Remember that, unlike kinetic energy, momentum is always conserved—it can't disappear.



So, there is a steady flow *in the z -direction* of x -direction momentum. Furthermore, the left-hand side of the equation

$$F/A = \eta v_0 / d$$

is just this momentum flow rate. The right hand side is the coefficient of viscosity multiplied by the gradient in the z -direction of the x -direction velocity.

Viewed in this way, $F/A = \eta v_0 / d$ is a *transport equation*. It tells us that the rate of transport of x -direction momentum downwards is proportional to the rate of change of x -direction velocity in that direction, and the constant of proportionality is the coefficient of viscosity. And, we can express this slightly differently by noting that the rate of change of x -direction *velocity* is proportional to the rate of change of x -direction *momentum density*.

Recall that we mentioned earlier the so-called *kinetic* viscosity coefficient, $\nu = \mu / \rho$. Using that in the equation

$$F / A = \eta v_0 / d = \nu \rho v_0 / d ,$$

replaces the velocity gradient with a x -direction *momentum* gradient. To abbreviate a clumsy phrase, let's call the x -direction momentum density π_x , and the current of this in the z -direction $J_z(\pi_x)$. Then our equation becomes

$$J_z(\pi_x) = \nu \frac{d\pi_x}{dz}.$$

The current of π_x in the z -direction is proportional to how fast π_x is changing in that direction.

This closely resembles heat flowing from a hot spot to a cold spot: heat energy flows towards the place where there is less of it, “downhill” in temperature. The rate at which it flows is proportional to the temperature gradient, and the constant of proportionality is the thermal conductivity (see later). Here, the π_x momentum flow is analogous: it too flows to where there is less of it, and the kinetic viscosity coefficient corresponds to the thermal conductivity.

*Viscosity in Gases

Suppose now we repeat Newton's suggested experiment, the two parallel plates with one at rest the other moving at steady speed, but with *gas* rather than liquid between the plates.

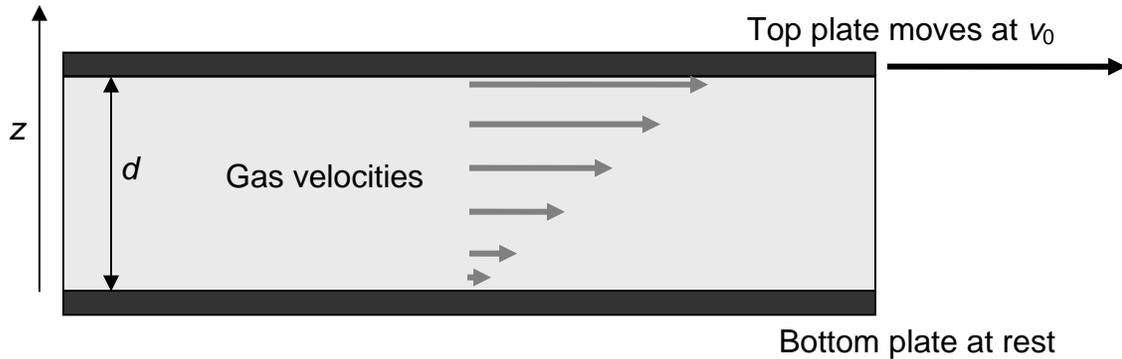
It is found experimentally that the equation $F / A = \eta v / d$ still describes the force necessary to maintain steady motion, but, not surprisingly, for gases anywhere near atmospheric pressure the coefficient of viscosity is far lower than that for liquids (not counting liquid helium—a special case):

Gas	Viscosity in 10^{-6} Pa.s
Air at 100K	7.1
Air at 300K	18.6
Air at 400K	23.1
Hydrogen at 300K	9.0
Helium at 300K	20.0
Oxygen at 300K	20.8
Nitrogen at 300K	17.9
Xenon at 300K	23.2

These values are from the CRC Handbook, 85th Edition, 6-201.

Note first that, in contrast to the liquid case, gas viscosity *increases* with temperature. Even more surprising, it is found experimentally that over a very wide range of densities, gas viscosity is *independent of the density* of the gas!

Returning to the two plates, and picturing the gas between as made up of layers moving at different speeds as before:



The first thing to realize is that at atmospheric pressure the molecules take up something like *a thousandth* of the volume of the gas. The previous picture of crowded molecules jostling each other is completely irrelevant! As we shall discuss in more detail later, the molecules of air at room temperature fly around at about 500 meters per second, the molecules have diameter around 0.35 nm, are around 3 or 4 nm apart on average, and travel of order 70 nm between collisions with other molecules.

So where does the gas viscosity come from? Think of two adjacent layers of gas moving at different speeds. Molecules from the fast layer fly into the slow layer, where after a collision or two they are slowed to go along with the rest. At the same time, some slower molecules fly into the fast layer. Even if we assume that kinetic energy is conserved in each individual molecular collision (so we're ignoring for the moment excitation of internal modes of the molecules) the *macroscopic* kinetic energy of the layers of gas decreases overall (see the exercise in the preceding section). How can that be? Isn't energy conserved? Yes, total energy *is* conserved, what happens is that some of the macroscopic kinetic energy of the gas moving as a smooth substance has been transferred into the microscopic kinetic energy of the individual molecules moving in random directions within the gas, in other words, into the random molecular kinetic energy we call heat.

***Estimating the Coefficient of Viscosity for a Gas: Momentum Transfer and Mean Free Path**

The way to find the viscosity of a gas is to calculate the rate of z -direction (downward) transfer of x -momentum, as explained in the previous section but one.

The moving top plate maintains a steady horizontal flow pattern, the x -direction speed at height z

$$v(z) = v_0 z / d$$

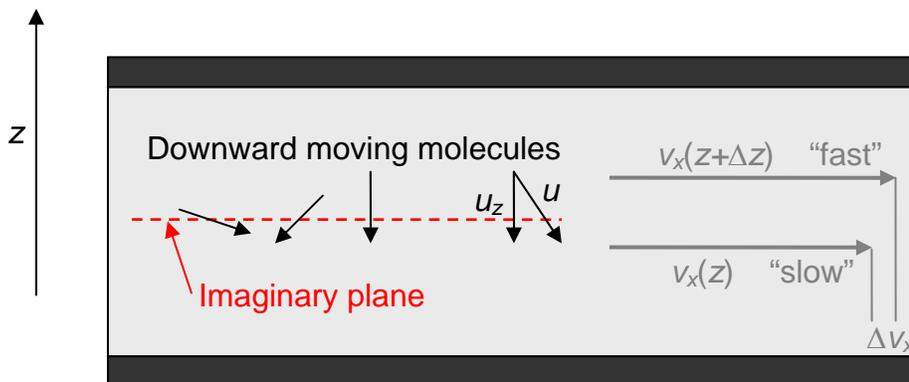
As explained above, the moving plate is feeding x -direction momentum into the gas at a rate $F / A = \eta v_0 / d$, this momentum moves down through the gas at a steady rate, and the coefficient of viscosity tells us what this rate of momentum flow is for a given velocity gradient.

In fact, this rate of momentum flow can be calculated from a simple kinetic picture of the gas: remember the molecules have about a thousand times more room than they do in the liquid state, so the molecules go (relatively) a long way between collisions. We shall examine this kinetic picture of a gas in much more detail later in the course, but for now we'll make the simplifying assumption that the molecules all have speed u , and travel a distance l between collisions. Actually this approximation gets us pretty close to the truth.

We take the density of molecules to be n , the molecular mass m . To begin thinking about x -direction momentum moving downwards, imagine some plane parallel to the plates and between them. Molecules from above are shooting through this plane and colliding with molecules in the slower moving gas below, on average transferring a little extra momentum in the x -direction to the slower stream. At the same time, some molecules from the slower stream are shooting upwards and will slow down the faster stream.

Let's consider first the molecules passing through the imaginary plane from above: we're only interested in the molecules already moving *downwards*, that's half of them, so a molecular density of $n/2$. If we assume for simplicity that all the molecules move at the same speed u , then the average *downward* speed of these molecules $\overline{u_z} = u/2$

($u_z = u \cos \theta$, and the average value of $\cos \theta$ over all downward pointing directions is $1/2$.)



Thus the number of molecules per second passing through the plane from above is

$$\frac{n}{2} \overline{|u_z|} = \frac{nu}{4}$$

and the same number are of course coming *up* from below! (Not shown.) (To see what's going on, the mean free path shown here is hugely exaggerated compared with the distance between the plates, of course.)

But this isn't quite what we want: we need to know how *efficiently* these molecules crossing the plane are transferring momentum for the fast moving streams above to the slower ones below. Consider one particular molecule going from the faster stream at some downward angle into the slower stream.

Let us assume it travels a distance l from its last collision in the "fast" stream to its first in the "slow" stream. The average distance between collisions is called the **mean free path**, here "mean" is used in the sense of "average", and is denoted by l . We are simplifying slightly by taking all distances between collisions to be l , so we don't bother with statistical averaging of the distance traveled. This does not make a big difference. The distance traveled *in the downward direction* is $\Delta z = lu_z / u$, so the (x -direction) speed difference between the two streams is

$$\Delta v = \frac{dv(z)}{dz} \Delta z = \frac{v_0}{d} \cdot \frac{lu_z}{u}.$$

The molecule has mass m , so *on average* the momentum transferred from the fast stream to the slow stream is $\Delta p = m\Delta v$. With our simplifying assumption that all molecules have the same speed u , all downward values of u_z between 0 and u are equally likely, and the density of downward-moving molecules is $n/2$, so the rate of transfer of momentum by downward-moving molecules through the plane is

$$\frac{n}{2} \overline{u_z} m \Delta v = \frac{nm}{2} \cdot l \cdot \frac{\overline{u_z^2}}{u} \cdot \frac{v_0}{d}.$$

At the same time, molecules are moving upwards from the slower streams into the faster ones, and the calculation is exactly the same. These two processes have the same sign: in the first case, the slower stream is gaining forward momentum from the faster, in the second, the faster stream is losing forward momentum, and in both cases total forward momentum is conserved. Therefore, the two processes make the same contribution, and the total momentum flow rate (per unit area) across the plane is

$$\text{momentum transfer rate} = nm \cdot l \cdot \frac{\overline{u_z^2}}{u} \cdot \frac{v_0}{d} = \frac{1}{3} nmlu \cdot \frac{v_0}{d}$$

using $\overline{u_z^2} = u^2 / 3$.

Evidently this rate of downward transfer of x -direction momentum doesn't depend on what level between the plates we choose for our imaginary plane, it's the same momentum flow all the way from the top plate to the bottom plate: so it's simply the rate at which the moving top plate is supplying x -direction momentum to the fluid,

$$\text{momentum supply rate} = F / A = \eta v_0 / d.$$

Since the momentum supplied moves steadily downwards through the fluid, the supply rate *is* the transfer rate, the two equations above are for the same thing, and we deduce that the coefficient of viscosity

$$\eta = \frac{1}{3} nmlu.$$

It should be noted that the u in the above equation is the mean speed of the atoms or molecules, which is about 8% greater than the rms speed appearing in the formula for gas pressure. In deriving the formula, we *did* make the simplifying assumption that all the molecules move at the same speed, and had the same mean free path, but Maxwell showed in 1860 that a more careful treatment gives essentially the same result. As discussed below, this result was historically important because when put together with other measurements, in particular the van der Waals isotherms for real gases, *it led to estimates of the sizes of atoms*. A great deal of effort has been expended improving Maxwell's formula. Maxwell assumed that after a collision, the molecules recoiled in random directions. In fact, a fast-moving molecule will tend to be deflected less, so fast-moving downward molecules will have effectively longer mean free paths than slow-moving ones, and this will substantially increase the viscosity over that predicted using the "slow" mean free path. Further progress can be made if the interaction between molecules is represented by a definite potential, so the scattering can be analyzed in detail. From measurements of viscosity as a function of density, it is then possible to learn something about the actual intermolecular potential.

***Why Doesn't the Viscosity of a Gas Depend on Density?**

Imagine we have a gas made up of equal numbers of red and green molecules, which have the same size, mass, etc. One of the molecules traveling through will on average have half its collisions with red molecules, half with green. If now all the red molecules suddenly disappear, the collision rate for our wandering molecule will be halved. This means its mean free path l will double. So, since the coefficient of viscosity $\eta = \frac{1}{3} nmlu$, halving the gas number density n at the same time doubles the mean free path l , so η is *unchanged*. (η does finally begin to drop when there is so little gas left that the mean free path is of order the size of the container.)

Another way to see this is to think about the molecules shooting down from the faster stream into slower streams in the two-plate scenario. If the density is halved, there will only be half the molecules moving down, but each will deliver the x -momentum difference twice as far—the further they go, the bigger the x -velocity difference between

where they begin and where they end, and the more effective they are in transporting x-momentum downwards.

*Comparing the Viscosity Formula with Experiment

We'll repeat the earlier table here for convenience:

Gas	Viscosity in 10^{-6} Pa.s
Air at 100K	7.1
Air at 300K	18.6
Air at 400K	23.1
Hydrogen at 300K	9.0
Helium at 300K	20.0
Oxygen at 300K	20.8
Nitrogen at 300K	17.9
Xenon at 300K	23.2

These values are from the CRC Handbook, 85th Edition, 6-201.

It's easy to see one reason why the viscosity increases with temperature: from $\eta = (1/3)nmlu$, η is proportional to the average molecular speed u , and since this depends on temperature as $\frac{1}{2}m\overline{u^2} = \frac{3}{2}k_B T$ (if this is unfamiliar, be assured we'll be discussing it in detail later), this factor contributes a \sqrt{T} dependence. In fact, though, from the table above, the increase in viscosity with temperature is more rapid than \sqrt{T} . We know the density and mass remain constant (we're far from relativistic energies!) so if the analysis is correct, the mean free path must also be increasing with temperature. In fact this is what happens—many of the changes of molecular direction in flight are not caused by hard collisions with other molecules, but by longer range attractive forces (van der Waals forces) when one molecule simply passes reasonably close to another. Now these attractive forces obviously act for a shorter time on a faster molecule, so it is deflected less. This means that as temperature and molecular speed increase, the molecules get further in approximately the same direction, and therefore transport momentum more effectively.

Comparing viscosities of different gases, at the same temperature and pressure they will have the same number density n . (Recall that a mole of any gas has Avogadro's number of molecules, 6×10^{23} , and that at standard temperature and pressure this occupies 22.4 liters, for any gas, if tiny pressure and volume corrections for molecular attraction and molecular volume are ignored.) The molecules will also have the same kinetic energy since they're at the same temperature (see previous paragraph).

Comparing hydrogen with oxygen, for example, the molecular mass m is up by a factor 16 but the velocity u decreases by a factor 4 (since the molecular kinetic energies $\frac{1}{2}mv^2$ are the same at the same temperature). In fact (see the table above), the viscosity of

oxygen is only *twice* that of hydrogen, so from $\eta = \frac{1}{3}nmlu$ we conclude that the hydrogen molecule has twice the mean free path distance l between collisions—not surprising since it is smaller. Also, helium must have an even longer mean free path, again not surprising for the smallest molecule in existence. The quite large difference between nitrogen and oxygen, next to each other in the periodic table, is because N_2 has a trivalent bonding, tighter than the divalent O_2 bonding, and in fact the internuclear distance in the N_2 molecule is 10% less than in O_2 . Xenon is heavy (atomic weight 131), but its mean free path is shorter than the others because the atom is substantially larger, and so an easier target.

Historically, the sizes of many atoms and molecules were first estimated from viscosity measurements using this method. In fact, just such a table of atomic and molecular diameters, calculated on the assumption that the atoms or molecules are hard spheres, can be found in the CRC tables, 85th edition, 6-47. But the true picture is more complicated: the atoms are not just hard spheres, as mentioned earlier they have van der Waals attractive forces between them, beyond the outermost shell. Actually, the electronic densities of atoms and molecules can now be found fairly precisely by quantum calculations using self-consistent field methods, and the resulting “radii” are in rough agreement with those deduced from viscosity (there is no obvious natural definition of radius for the electron cloud).

More General Laminar Flow Velocity Distributions

We’ve analyzed a particularly simple case: for the fluid between two parallel plates, the bottom plate at rest and the top moving at steady speed v_0 , the fluid stream velocity increases *linearly* from zero at the bottom to v_0 at the top. For more realistic laminar flow situations, such as that away from the banks in a wide river, or flow down a pipe, the rate of velocity increase on going from the fixed boundary (river bed or pipe surface) into the fluid is no longer linear, that is, $dv(z)/dz$ is not the same everywhere.

The key to analyzing these more general laminar flow patterns is to find the forces acting on a small area of one of the “sheets”. (Or a larger area of sheet if the flow is uniform in the appropriate direction.) There will be external forces such as gravity or pressure maintaining the flow, which must balance the viscous drag forces exerted by neighboring sheets if the fluid is not accelerating. The sheet-sheet drag force is equivalent to the force $F/A = \eta v_0/d$ exerted by the top “sheet” of fluid on the top plate in the previous discussion, except that now the forces are within the fluid. As we discussed earlier, the drag force on the top plate comes from molecules close to the plate bouncing off, gaining momentum, which is subsequently transferred to other molecules a little further away. For liquids, this mechanism involves distances of order a few molecular diameters, for gases a few mean free paths. In either case, the distances are tiny on a macroscopic scale. This means that the appropriate formula for the drag force on a plate, or that between one sheet of fluid on another, is

$$\frac{F}{A} = \eta \frac{dv(z)}{dz}.$$

The *rate of change of stream velocity close to the interface* determines the drag force.

We shall show in the next lecture how this formula can be used to determine the flow pattern in a river, and that in a circular pipe.

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