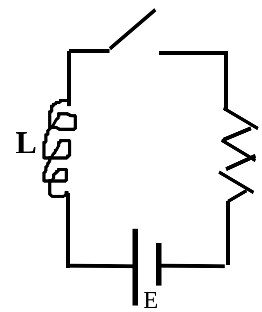
Oscillations in Circuits

The example above leads us directly to the case of a circuit with bot L & C in it.



Before we consider this let discuss the simpler circuit with just L & R.

Applying the loop rule on the above circuit we can write:

$E - L\,\frac{dI}{dt} - IR = 0$. This is a differential equation in I with the independent variable as t

Try the solution to this equation as:

$I \sim e^{-t(L/R)} + C$ therefore $\frac{dI}{dt} = -\left(\frac{R}{L}\right) e^{-t/\left(\frac{L}{R}\right)}$. Hence the differential equation becomes

$$\frac{E}{R} - \frac{L}{R}\left(\frac{-R}{L}\right)e^{-\frac{t}{(L/R)}} - e^{-\frac{t}{(L/R)}} - c = 0 \quad \Rightarrow \quad c = \left(\frac{E}{R}\right)$$

and

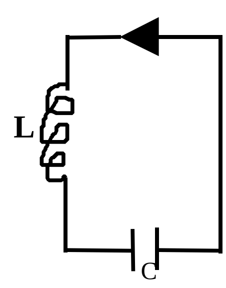
$$I = \frac{E}{R}e^{-t/(L/R)} \quad \rightarrow \quad \text{decay of current}.$$

$$I = \frac{E}{R}\left(1 - e^{-t/\left(\frac{L}{R}\right)}\right) \quad \text{for charging}.$$

$t \sim RC$

Note the parallel's between RC & RL circuits. In RC circuits the time constant $t \sim RC$ and in RL circuits $t \sim \left(\frac{L}{R}\right)$.

Oscillations: Circuit w/ both L & C.



Count voltage in the direction of current

$$-L\,\frac{dI}{dt} - \frac{Q}{C} = 0$$

How to solve this equation:

$$\frac{Q}{C} + L\frac{d^2Q}{dt^2} = 0$$

This equation reminds us of S.H.M. $k\,x + m\dfrac{d^2x}{dt^2} = 0$

Therefore $w = \sqrt{\dfrac{k}{m}}$ and $x = x_0 \, Cos(wt + \phi)$. Therefore $w = \sqrt{\dfrac{1}{LC}}$

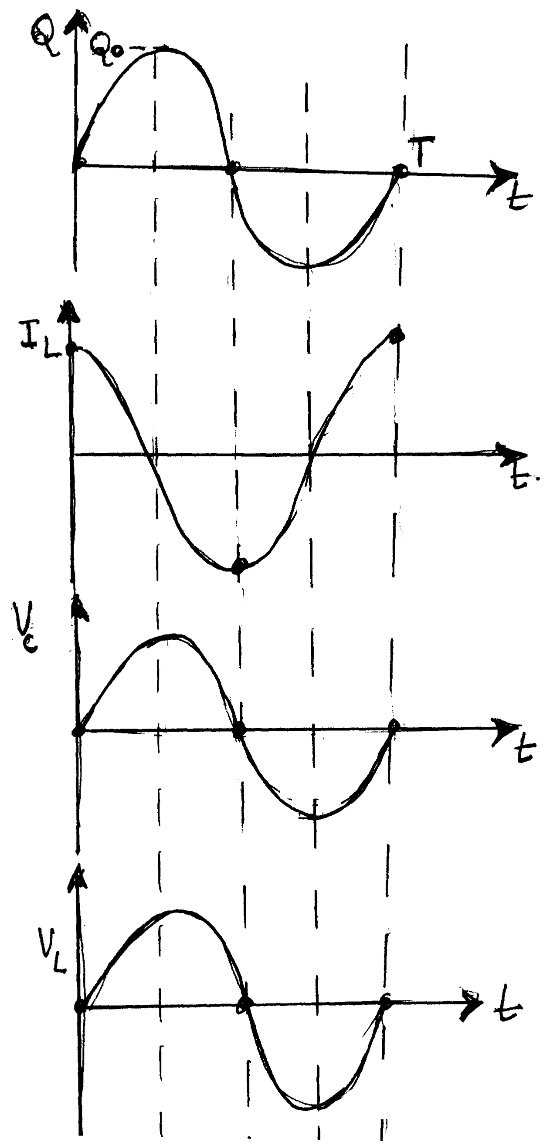
The charge on the capacitor (i.e. analog of x) oscillates:

$Q = Q_0 \, Cos(wt + \phi)$; Q_0 and ϕ are determined by initial conditions.

Also, $I = dQ/dt = Q_0 \, w \, Sin(wt + \phi)$

Or $I = I_0 \, Sin\,(wt + \phi)$ $\rightarrow$ same frequency.

Physical picture: Start with capacitor charged one way and close switch - current flows into L & flows to neutralize charges on capacitor - when V is zero I has reached a maximum value - I decays ⇒ EMF is opposite ⇒ charge flows in the same direction.



Since L & C are in parallel the voltages V_L & V_C are same.

Energy in the oscillating circuits: (No resistance):

$Q = Q_0 \, Cos\, wt$

$$I = \frac{dQ}{dt} = -w \, Q_0 Sin wt$$

But energy in a capacitor is:

$$U_C = \frac{Q^2}{2C} = \frac{Q_0^2}{2C} Cos^2 wt \, ; \; U_C^{max} = \frac{Q_0^2}{2C}$$

The energy in the inductor is:

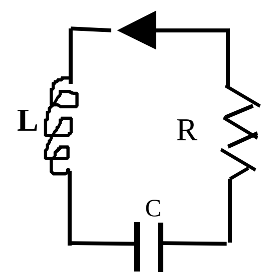
$$U_L = \frac{1}{2} L I^2 = \frac{1}{2} L w^2 Q_0^2 \, Sin^2 wt$$

Total $U = U_C + U_L$ or

$$U = \frac{Q_0^2}{2C}(Cos^2 wt + Sin^2 wt) = \frac{Q_0^2}{2C}$$

<u>Damped Oscillations</u>: Add a resistor → real world circuits always have them:

$$- L \frac{dI}{dt} = I R - \frac{Q}{C} = 0 .$$ because of currrent going ← positive charges are flowing into C.



Similar to damped mechanical oscillator with drag force i.e.

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 .$$

Solution is: $Q = Q_0 e^{-at} Cos(w't + f)$. Substituting this we find $a = \frac{R}{2L}$ and

$$w'^2 = \frac{1}{LC} - \frac{R^2}{4L^2} = w^2 - a^2$$

Note that something happens when $w^2 - a^2$ i.e. when $R = R_C = 2\sqrt{\frac{L}{C}}$ the circuit is damped or there is "critical damping".