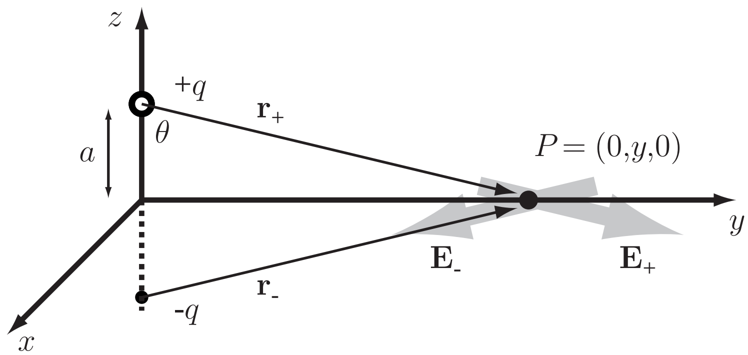
Homework #2 Solutions
Due: Friday September 11, 1998

1.



Assume for now that our field point is confined to the y-axis. The electric field due to the $+q$ and $-q$ charges are shown in the figure and labeled $\mathbf{E}_+$ and $\mathbf{E}_-$, respectively. By the **principle of superposition**, the total field is the sum $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-$. The form for $\mathbf{E}_\pm$ is:

$$\mathbf{E}_\pm = \frac{\pm q \mathbf{r}_\pm}{4\pi\epsilon_0 r_\pm^3}, \tag{1}$$

where $r_\pm \equiv y\mathbf{j} \mp a\mathbf{k}$. By our choice of field point, there is no field component in the x-direction. For the y-component of eq. (1),

$$E_y = E_{+y} + E_{-y} = \frac{1}{4\pi\epsilon_0}\left(\frac{(+q)y}{(y^2+a^2)^{3/2}} + \frac{(-q)y}{(y^2+a^2)^{3/2}}\right) = 0. \tag{2}$$

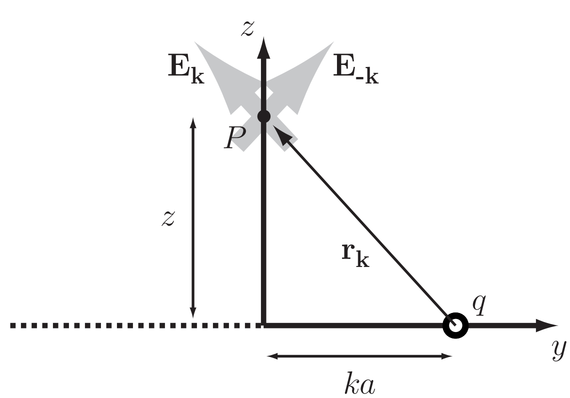
Note that each term is the same as the magnitude of the field $1/4\pi\epsilon_0 r^2$ multiplied by $\sin\theta$, with θ as shown in the figure. One could write this down by inspection, stating that the y-components must be equal and opposite by symmetry. The z-component is a different story,

$$E_z = E_{+z} + E_{-z} = \frac{1}{4\pi\epsilon_0}\left(\frac{(+q)(-a)}{(y^2+a^2)^{3/2}} + \frac{(-q)(+a)}{(y^2+a^2)^{3/2}}\right) = \frac{-2qa}{4\pi\epsilon_0 (y^2+a^2)^{3/2}}. \tag{3}$$

We could just as well have calculated the solution on the x-axis, so we could make the substitution $y \to x$ in the above equation, or even better, $y \to \sqrt{x^2+y^2}$, taking care of the entire xy-plane. The answer for the field in the $z = 0$ plane is then

$$\mathbf{E} = \frac{-2qa\mathbf{k}}{4\pi\epsilon_0 (x^2+y^2+a^2)^{3/2}}. \tag{4}$$

2.



Ignore, for the moment, all except one of the charges, and try to not be distracted by the fact that there are supposed to only be a finite number of charges. The vector connecting a charge q located at $(0, ka, 0)$, where k is some integer, with a field point P located on the z-axis is given by $\mathbf{r_k} = -ka\mathbf{j} + z\mathbf{k}$. The electric field at $(0, 0, z)$ due to this charge is

$$\mathbf{E_k} = \frac{q\mathbf{r_k}}{4\pi\epsilon_0 r_k^3} = \frac{q(-ka\mathbf{j} + z\mathbf{k})}{4\pi\epsilon_0(k^2a^2 + z^2)^{3/2}}. \tag{5}$$

This expression works for all integer k. We can use the principle of superposition to obtain the field at $(0, 0, z)$ due to a finite number of charges, assuming that we add them up correctly:

$$\mathbf{E} = \sum_{k=-\ell}^{\ell} \mathbf{E_k} = \frac{q}{4\pi\epsilon_0} \sum_{k=-\ell}^{\ell} \frac{-ka\mathbf{j} + z\mathbf{k}}{(k^2a^2 + z^2)^{3/2}}. \tag{6}$$

Remember that the problem states that there are n charges, where n is odd and the central charge is located at the origin. This implies that we are performing the sum for n charges when $\ell \equiv (n-1)/2$, which is also an integer. Now let us take a close look at eq. (6). When we add up the terms from $k = -\ell$ to $k = +\ell$, we see that the y-components will cancel out. This could have been deduced from the beginning, as this is also clear from the symmetry of the problem. We are therefore left with only a z-component to the electric field for field points on the z-axis:

$$\mathbf{E} = \frac{qz\mathbf{k}}{4\pi\epsilon_0} \sum_{k=-\ell}^{\ell} \frac{1}{(k^2a^2 + z^2)^{3/2}}. \tag{7}$$

To find the field in the limit $n \to \infty$, we should write the sum in a standard form. For the curious, one can most easily define "standard form" to mean a form of an equation one is most likely to encounter in some standard math tables or such. Generally this involves putting the term that is summed into unitless factors, which is a good idea in any case. To wit:

$$\mathbf{E} = \frac{q\mathbf{k}}{4\pi\epsilon_0 z^2} \sum_{k=-\ell}^{\ell} \frac{1}{\left[1 + \left(\frac{a}{z}\right)^2 k^2\right]^{3/2}}. \tag{8}$$

The field in the limit $n \to \infty$ is found by replacing the occurrences of "ℓ" in the above equation with "∞". A not-so-quick perusal of the series tables will no doubt greet the reader, as it did us, with disappointment. Although the series clearly converges using an elemental calculus test, it cannot be evaluated (at least in a straightforward manner) as the author of the problem would probably claim.

3 (Tipler 18-32). The force on the electron in a field is given by $\mathbf{F} = (-e)\mathbf{E}$. Between the plates the electric field is constant and is given by $\mathbf{E} = E_0\mathbf{j}$, where $E_0 = 3.5 \times 10^3$ V/m. From what we know about the electric potential, this means that there are $(3.5 \times 10^3 \text{ V/m}) \times (0.02 \text{ m}) = 70$ V between the plates (which is not hard to generate with a power supply). A constant electric field translates to a constant force, which causes a constant acceleration. The motion of the electron is then given by

$$\mathbf{r} = \mathbf{r_0} + \mathbf{v_0}t + \frac{1}{2}\mathbf{a}t^2, \tag{9}$$

where the acceleration $\mathbf{a} = \mathbf{F}/m = -e\mathbf{E}/m$. To get our final answer, we could just take the solutions for an object that experiences constant acceleration due to gravity and make the substitution $g \to eE_0/m$. We will not be doing this here, however, because we'd like to check the possibility that the electron could hit the top plate, even though the force is down.

The conditions which fit this problem are $\mathbf{v_0} = v_0(\mathbf{i} + \mathbf{j})/\sqrt{2}$ and $\mathbf{r_0} = 0$ (we'll define the origin as the point where the electron starts). Eq. (9) can be written in parametric form:

$$x = \frac{v_0}{\sqrt{2}}t, \qquad y = \frac{v_0}{\sqrt{2}}t - \frac{eE_0}{2m}t^2. \tag{10}$$

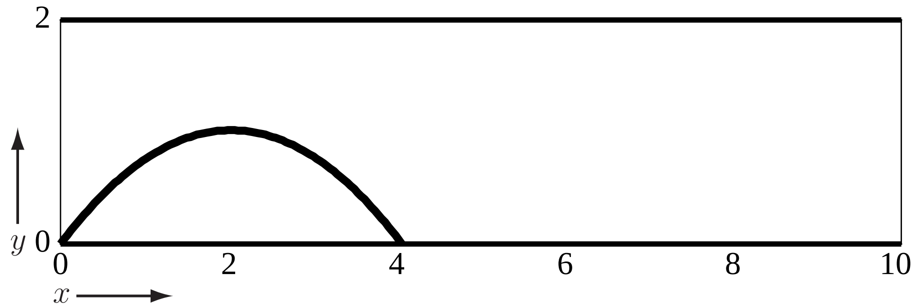
We can use the quadratic formula to solve the y equation for t, to obtain the times when the electron hits a plate:

$$t = \frac{mv_0}{eE_0\sqrt{2}} \pm \sqrt{\frac{m^2v_0^2}{2e^2E_0^2} - \frac{2my}{eE_0}}. \tag{11}$$

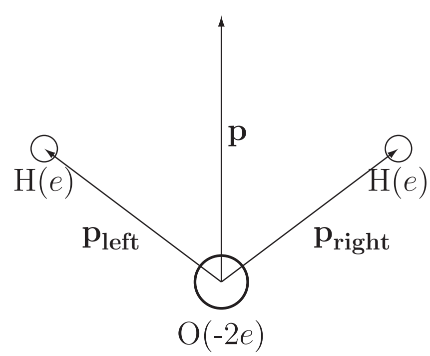
To check that the electron does not hit the top plate, plug $y = 2$ cm into the above equation. The fact that the quantity under the square root is negative for this value, a non-physical answer, shows that the electron never reaches a height of 2 cm. The electron is at the bottom plate (where $y = 0$) for times $t_1 = 0$ and $t_2 = 1.15 \times 10^{-8}$ s, so the electron hits a distance

$$x_2 = \frac{v_0}{\sqrt{2}}t_2 = 4.1\,\mathrm{cm} \tag{12}$$

to the right of its starting point. The trajectory of the electron between the plates is traced in the following figure.



4 (Tipler 18-42).



There are two dipoles here: one that points from the one of the charges ($-e$) at the origin to the left proton of charge ($+e$) and one that points from the other charge ($-e$) at the origin to the right proton. This way both charges at the origin get "used up." The total dipole moment is the vector sum of these two:

$$\mathbf{p} = \mathbf{p_{left}} + \mathbf{p_{right}} = e(-x_0\mathbf{i} + y_0\mathbf{j}) + e(x_0\mathbf{i} + y_0\mathbf{j}) = 2ey_0\mathbf{j}, \tag{13}$$

where $x_0 = 0.077$ nm and $y_0 = 0.058$ nm. The dipole moment is then of magnitude 1.9×10^{-29} C·m, and points from the oxygen nucleus to the midpoint of the line adjoining the two hydrogen nuclei.

5 (Tipler 19-15). (a) The surface area of the sphere is $A = 4\pi R^2 = 4\pi(0.5)^2 = 3.14$ m^2.
(b) The electric field on the surface of the sphere is $E = kq/R^2 = (9\times10^9)(2\times10^{-6})/(0.5)^2 = 7.2\times10^4$ V/m.
(c) The flux is $\Phi = EA = 2.3 \times 10^5$ V·m.
(d) Although the electric field at all points on the sphere would change when the point charge is moved from the center, and although the field is no longer uniform on the surface, according to Gauss's law the flux through the entire surface is the same because it encloses the same charge.
(e) The field lines that go through the sphere must, in addition, go through the cube since there is no charge in the intervening space. The flux through the cube is therefore the same as that found in part (c).