Homework #3 Solutions
Due: Friday September 18, 1998

1. The electric field at point A due to a bit of charge dq located a distance x away is

$$dE_A = \frac{dq}{4\pi\epsilon_0 x^2}. \tag{1}$$

In this case we have $dq = \lambda\, dx$ with $\lambda = (3 \times 10^{-9}\ \mathrm{C})/(0.1\ \mathrm{m})$, so the total field at A is given by

$$E_A = \int dE_A = \frac{\lambda}{4\pi\epsilon_0}\int_{0.3\,\mathrm{m}}^{0.13\,\mathrm{m}} \frac{dx}{x^2} = \frac{\lambda}{4\pi\epsilon_0}\left(-\frac{1}{x}\right|_{0.3}^{0.13} = 6900\,\mathrm{V/m}. \tag{2}$$

At point B, we need to do the same integral but the distance has changed, and we need to take only the vertical component (the horizontal component cancels when summed over the whole rod):

$$dE_B = \frac{y_0\, dq}{4\pi\epsilon_0 (x^2 + y_0^2)^{3/2}}, \tag{3}$$

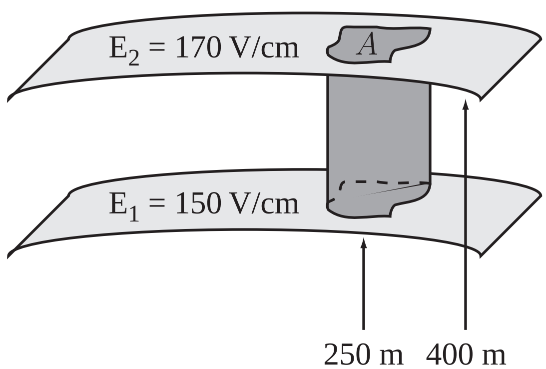
where we take $y_0 \equiv 0.05$ m and x is the relative horizontal displacement. The total field is

$$E_B = \frac{y_0\lambda}{4\pi\epsilon_0}\int_{-0.05}^{0.05} \frac{dx}{(x^2+y_0^2)^{3/2}} = \frac{2y_0\lambda}{4\pi\epsilon_0}\left(\frac{x}{y_0^2\sqrt{x^2+y_0^2}}\right|_0^{0.05} = 7600\,\mathrm{V/m}. \tag{4}$$

The following integral from the standard tables was used:

$$\int \frac{dx}{(ax^2+b)^{3/2}} = \frac{x}{b\sqrt{ax^2+b}}. \tag{5}$$

2 (Tipler 19-29).



The fact that the electric field is in the vertical direction at both 250 m and 400 m, and that we should assume that the volume between these altitudes contains a uniform charge density, indicates that the electric field should be vertical inside this region as well. One can then solve this problem by applying Gauss's law to a pillbox with one face at 250 m, and the other face at 400 m.

$$\begin{aligned}\oint_S \mathbf{E}\cdot d\mathbf{A} &= (\text{flux through top side}) + (\text{flux through bottom side}) + (\text{flux through sides}), && (6)\\ &= (-E_2 A) + (E_1 A) + 0, && (7)\\ &= (E_1 - E_2)A, && (8)\end{aligned}$$

where we have defined E_1 = (field at 250 m) = 150 V/m and E_2 = (field at 400 m) = 170 V/m. The first term sum above is negative because the field is pointing in the opposite direction of the pillbox surface normal. The other part of Gauss's law that we need to evaluate is:

$$\frac{1}{\epsilon_0}\int_V \rho\, dV = \frac{\rho A \Delta h}{\epsilon_0}, \tag{9}$$

where $\Delta h =$ (400 m)-(250 m) is the difference in heights. Since these two quantities must be equal, we have

$$\rho = \frac{\epsilon_0(E_1 - E_2)}{\Delta h} = -1.2 \times 10^{-12}\ \mathrm{C/m^3}. \tag{10}$$

Note that instead of neglecting the curvature of the earth, we could have explicitly taken it into account. Start by assuming the earth is a big ball of charge, wherein,

$$E_1 = \frac{Q_1}{4\pi\epsilon_0 R_1^2}. \tag{11}$$

Here $R_1 \equiv R_{\text{earth}} + h_1$. Hence the charge inside is given by $Q_1 = 4\pi\epsilon_0 E_1 R_1^2$, and likewise $Q_2 = 4\pi\epsilon_0 E_2 R_2^2$. The difference in charge $\Delta Q = Q_2 - Q_1$ must be the charge in between the two heights, and the charge density (assuming it to be uniform) must be given by $\rho = \Delta Q/\Delta V$, where $\Delta V = \frac{4}{3}\pi(R_2^3 - R_1^3)$. This value, as you will find, is exactly the same (within our accuracy) as was found when neglecting the curvature of the earth. The reason why, of course, is because the size of the earth is so big compared to the difference in the altitudes of interest.

3 (Tipler 19-48). (*a*) The total charge Q in the sphere is given by:

$$Q = \int_V \rho\, dV = \int_0^R (Ar)(4\pi r^2\, dr) = 4\pi A \int_0^R r^3\, dr = \pi A R^4. \tag{12}$$

(*b*) We know through the numerous examples we've done that the field due to a spherical shell is zero inside the shell, and that outside the shell it behaves like a point charge located at its center. This can be succinctly derived using Gauss's law:

$$\oint_S \mathbf{E}\cdot d\mathbf{A} = \frac{1}{\epsilon_0}\int_V \rho\, dV, \tag{13}$$

$$E_r \cdot 4\pi r^2 = \frac{Q_{\text{inside}}}{\epsilon_0}, \tag{14}$$

$$\Downarrow$$

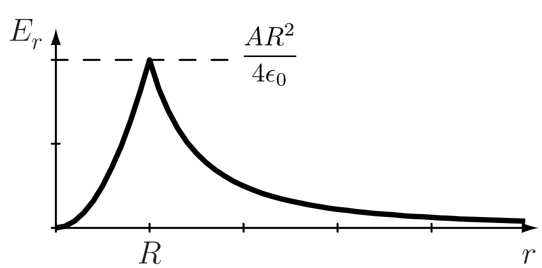
$$E_r = \frac{Q_{\text{inside}}}{4\pi\epsilon_0 r^2}. \tag{15}$$

Since the quantity

$$Q_{\text{inside}} = \begin{cases} \pi A r^4 & r < R \\ \pi A R^4 & r > R \end{cases}, \tag{16}$$

we have

$$E_r = \begin{cases} \dfrac{Ar^2}{4\epsilon_0} & r < R \\ \dfrac{AR^4}{4\epsilon_0 r^2} & r > R \end{cases}. \tag{17}$$



4 (Tipler 20-38). If we call the energy of the emitted alpha particle $B = 5.30$ MeV, then the closest it could have been to the nucleus can be found by equating B to the potential energy at some value R. This would be the point at which all of its kinetic energy is converted into the potential energy between the alpha and the nucleus. So,

$$B = \frac{q_1 q_2}{4\pi\epsilon_0 R} \qquad \Rightarrow \qquad R = \frac{q_1 q_2}{4\pi\epsilon_0 B}. \tag{18}$$

For $q_1 = 2e$ and $q_2 = 82e$, we obtain $R = 4.5 \times 10^{-14}$ m or 45 fm.

5 (Tipler 20-62). The fact that the inner shell is grounded gives it a source of charge, which will flow onto or off of the shell to force it to zero potential. The potential inside the inner shell is therefore also zero, so that we have $V(r < a) = 0$. Outside of the outer shell, we know from the result of Problem 3 that the field

$$E_r(r > b) = \frac{Q_{\text{inside}}}{4\pi\epsilon_0 r^2}, \tag{19}$$

so that the potential must be

$$V(r > b) = \frac{Q_{\text{inside}}}{4\pi\epsilon_0 r} = \frac{q + Q}{4\pi\epsilon_0 r}, \tag{20}$$

where we have set q to be the charge on the inner shell. For $a < r < b$, we know that the outer shell makes no contribution to the field, and also that the field depends on the charge on the inner shell as

$$E_r(a < r < b) = \frac{q}{4\pi\epsilon_0 r^2}. \tag{21}$$

This means the potential must be given by

$$V(a < r < b) = \frac{q}{4\pi\epsilon_0 r} + C, \tag{22}$$

where C is some constant. The reader may be curious as to why there is a constant of integration here, when often it seems there isn't one. The best way to deal with this kind of situation is to assume there is *always* an integration constant, which can be zero if there is a compelling reason. Here, for example, there is an extremely good reason why $C \neq 0$, namely that we wouldn't be able to do the problem otherwise (i. e. it would be wrong). We can evaluate the value of C by applying the constraint that the electric potential be continuous. We know that the potential at $r = b$ must be $V(b) = (q + Q)/4\pi\epsilon_0 b$ from eq. (20), and this must be the value of eq. (22) as well:

$$\frac{q + Q}{4\pi\epsilon_0 b} = \frac{q}{4\pi\epsilon_0 b} + C, \qquad \Rightarrow \qquad C = \frac{Q}{4\pi\epsilon_0 b}. \tag{23}$$

Finally, we must also set $V(a) = 0$ in eq. (22),

$$V(a < r < b, r = a) = \frac{q}{4\pi\epsilon_0 a} + \frac{Q}{4\pi\epsilon_0 b} = 0. \tag{24}$$

Solve this equation to obtain the charge q on the inner sphere,

$$q = -\frac{a}{b} Q. \tag{25}$$

An alternative to the method outlined above would be to perform a definite integral of eq. (21) to find the potential at $r = b$:

$$V(b) - V(a) = -\frac{q}{4\pi\epsilon_0} \int_a^b \frac{dr}{r^2}. \tag{26}$$

Since $V(a) = 0$ and $V(b) = (q + Q)/4\pi\epsilon_0 b$, we have

$$\frac{q + Q}{b} = \frac{q}{b} - \frac{q}{a}, \tag{27}$$

which gives the same answer. The full expression for the potential is then

$$V(r) = \begin{cases} 0 & r < a \\ \dfrac{Q}{4\pi\epsilon_0 b}\left(1 - \dfrac{a}{r}\right) & a < r < b \\ \dfrac{Q}{4\pi\epsilon_0 r}\left(1 - \dfrac{a}{b}\right) & r > b \end{cases} \tag{28}$$