Homework #6 Solutions
Due: Friday October 16, 1998

1 (Tipler 23-52). I encourage you to come speak to me concerning the solution to this problem, if you are really curious. I've twice been given a solution that doesn't suck, yet I am remain heady with insolution. It is suggested that as a study aid in the subject of infinite resistor networks the student solve Tipler 23-57.

2 (Tipler 23-65). (a) The power supplied by the battery is given by $P = I\mathcal{E}$. Since this is a capacitor charging circuit, we know that the current is

$$I(t) = \frac{\mathcal{E}}{R} e^{-t/RC}. \tag{1}$$

Therefore,

$$P_{\text{battery}}(t) = \frac{\mathcal{E}^2}{R} e^{-t/RC}. \tag{2}$$

(b) The power dissipated in the resistor is $P = I^2 R$. Using eq. (1), we obtain

$$P_{\text{R}}(t) = \frac{\mathcal{E}^2}{R} e^{-2t/RC}. \tag{3}$$

(c) The energy stored in the capacitor is given by

$$U = \frac{Q(t)^2}{2C} = \frac{\left(\int_0^t I(t)\,dt\right)^2}{2C}. \tag{4}$$

Let's evaluate this integral, again using eq. (1),

$$Q(t) = \int_0^t I(t')\,dt' = \frac{\mathcal{E}}{R}\int_0^t e^{-t'/RC} = C\mathcal{E}\left(1 - e^{-t/RC}\right). \tag{5}$$

Yes, I know this is also given in the book. So we have

$$U = \frac{1}{2}C\mathcal{E}^2\left(1 - e^{-t/RC}\right)^2 = \frac{1}{2}C\mathcal{E}^2\left(1 - 2e^{-t/RC} + e^{-2t/RC}\right). \tag{6}$$

This gives the energy stored in the capacitor as a function of time, but we were asked the rate at which it is stored, so we need to take its derivative:

$$\frac{dU}{dt} = \frac{1}{2}C\mathcal{E}^2\left(\frac{2}{RC}e^{-t/RC} - \frac{2}{RC}e^{-2t/RC}\right) = \frac{\mathcal{E}^2}{R}\left(e^{-t/RC} - e^{-2t/RC}\right). \tag{7}$$

Alert readers will notice that this is the difference between the power coming out of the battery and the power dissipated in the resistor, $P_{\text{battery}}(t) - P_{\text{R}}(t)$.
(d) We could refer to the rate at which energy is being stored in the capacitor as the power into the capacitor, P_{C}. It has a maximum in time given by the zero in its time derivative:

$$\frac{dP_{\text{C}}}{dt} = \frac{d^2U}{dt^2} = \frac{\mathcal{E}^2}{R}\left(-\frac{1}{RC}e^{-t/RC} + \frac{2}{RC}e^{-2t/RC}\right) = 0. \tag{8}$$

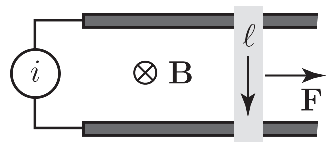
This expression is zero when

$$e^{-t/RC} = 2e^{-2t/RC}, \qquad \text{or} \qquad t = RC\ln 2. \tag{9}$$

The power P_{C} at this time is

$$P_{\text{C}}(t = RC\ln 2) = \frac{\mathcal{E}^2}{R}\left(e^{-\ln 2} - e^{-2\ln 2}\right) = \frac{\mathcal{E}^2}{R}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\mathcal{E}^2}{4R}. \tag{10}$$

3 (Tipler 24-50).



(a) The force on the bar is

$$\mathbf{F} = I\boldsymbol{\ell} \times \mathbf{B}. \tag{11}$$

According to the picture in the book, $\boldsymbol{\ell}$ points down, $\mathbf{B}$ points into the page (at right angles to $\boldsymbol{\ell}$), so $|\boldsymbol{\ell} \times \mathbf{B}| = B\ell$ and points to the right. By Newton's third law, the force to the right is

$$F = Ma = M\frac{dv}{dt} = BI\ell. \tag{12}$$

We can solve this differential equation by separating variables and integrating using the given initial conditions,

$$\int_0^v dv = \frac{BI\ell}{M}\int_0^t dt, \tag{13}$$

so the velocity is simply $v = (BI\ell/M)t$.
(b) The force is to the right, so the bar will move in that direction.
(c) The force due to the magnetic field must overcome the force due to static friction, which we know is given by $\mu_s N$, where N is the normal force exerted by the rails on the crossbar. There is no reason to think that $N \neq Mg$, so we must have that $I\ell B > \mu_s Mg$, or $B > \mu_s Mg/I\ell$.

Note that the fact that there is a *current* source attached instead of a voltage source is of particular importance in this problem.

4 (Tipler 24-42). (a) The book tells us that the Hall voltage is given by the expression

$$V_\mathrm{H} = \frac{IB}{net}, \tag{14}$$

and implies that the number density of the charge carriers is

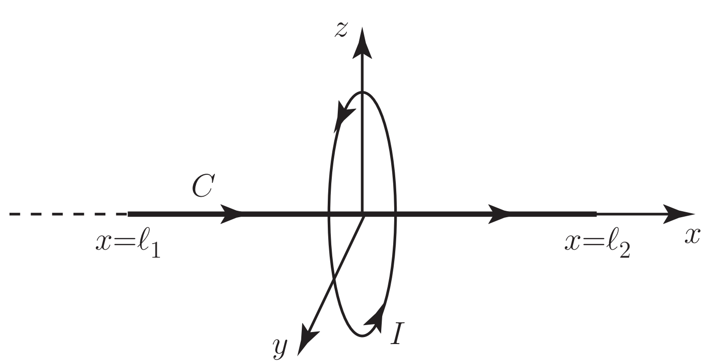
$$n = \frac{IB}{etV_\mathrm{H}}. \tag{15}$$

Since we are given all of the quantities on the right, we can punch up to numbers to get $2.4 \times 10^{23}\ \mathrm{cm}^{-3}$.
(b) The number density of atoms in beryllium is just a dimensional analysis problem:

$$n_\mathrm{B} = \frac{(1.83\ \mathrm{g}\cdot\mathrm{cm}^{-3})(6.022 \times 10^{23}\ \mathrm{atoms}\cdot\mathrm{mol}^{-1})}{9.01\ \mathrm{g}\cdot\mathrm{mol}^{-1}} = 1.2 \times 10^{23}\ \mathrm{atoms}\cdot\mathrm{cm}^{-3}. \tag{16}$$

(c) The number of free electrons per atom is just the number density of charge carriers (a) divided by the number density of atoms (b), which is 2, within our accuracy.

5 (Tipler 25-58).



(a) The magnetic field along the x-axis due to the current loop of radius a in the yz plane is given by

$$\mathbf{B} = \frac{\mu_0 I}{2} \frac{a^2\, \hat{\mathbf{i}}}{(a^2 + x^2)^{3/2}}. \tag{17}$$

We are asked to evaluate the line integral $\int \mathbf{B} \cdot d\boldsymbol{\ell}$ along a curve C that lies along the x-axis and goes from $x = -L$ to $x = L$. Therefore the curve C can be described by a vector $\boldsymbol{\ell} = x\,\hat{\mathbf{i}}$, giving $d\boldsymbol{\ell} = dx\,\hat{\mathbf{i}}$, so

$$\int_C \mathbf{B} \cdot d\boldsymbol{\ell} = \int_{-L}^{L} \mathbf{B} \cdot (dx\,\hat{\mathbf{i}}) = \frac{\mu_0 I a^2}{2} \int_{-L}^{L} \frac{dx}{(a^2 + x^2)^{3/2}} = \mu_0 I a^2 \int_0^L \frac{dx}{(a^2 + x^2)^{3/2}}. \tag{18}$$

This integral can be found in the tables (I use *Beta*, from CRC), so

$$\int_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I a^2 \left. \frac{x}{a^2\sqrt{a^2 + x^2}} \right|_0^L = \mu_0 I \frac{L}{\sqrt{a^2 + L^2}}. \tag{19}$$

(b) The limit of the line integral for very large L can be found by looking at just the fraction on the right, where we see that we can neglect the a^2 term (since it is small compared to the very large L^2),

$$\lim_{L \to \infty} \frac{L}{\sqrt{a^2 + L^2}} = \frac{L}{\sqrt{L^2}} = 1. \tag{20}$$

In the limit $L \to \infty$, the line integral is just $\mu_0 I$.

If you understand the part of the problem that tells you how this result can be related to Ampère's law, where the curve C is closed at $x = \pm\infty$ with a really big semicircle, CONGRATULATIONS! You *may* have a future in theoretical physics or mathematics in the field of complex analysis, where their motto is, "We work hard to make a better batwing."