Time Dilation: A Worked Example

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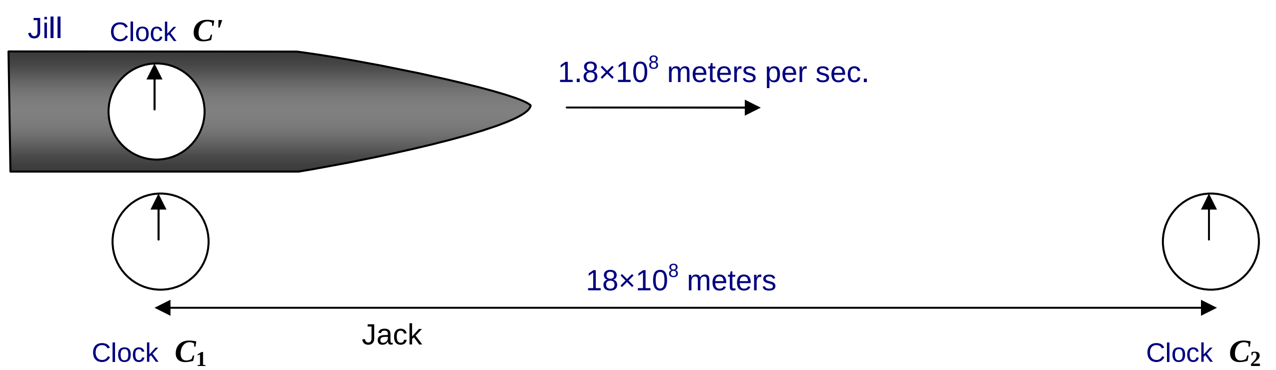
"Moving Clocks Run Slow" *plus* "Moving Clocks Lose Synchronization" *plus* "Length Contraction" leads to consistency!

The object of this exercise is to show explicitly how it is possible for two observers in inertial frames moving relative to each other at a relativistic speed to each see the other's clocks as running slow and as being unsynchronized, and yet if they both look at the same clock at the same time from the same place (which may be far from the clock), they will *agree* on what time it shows!

Suppose that in Jack's frame we have two synchronized clocks C_1 and C_2 set 18 x 10^8 meters apart (that's about a million miles, or 6 light-seconds). Jill's spaceship, carrying a clock C', is traveling at 0.6c, that is 1.8 x 10^8 meters per second, parallel to the line C_1C_2, passing close by each clock.



Jill in her relativistic rocket passes Jack's first clock at an instant when both their clocks read zero.

Suppose C' is synchronized with C_1 as they pass, so both read zero.

As measured by Jack the spaceship will take just 10 seconds to reach C_2, since the distance is 6 light seconds, and the ship is traveling at 0.6c.

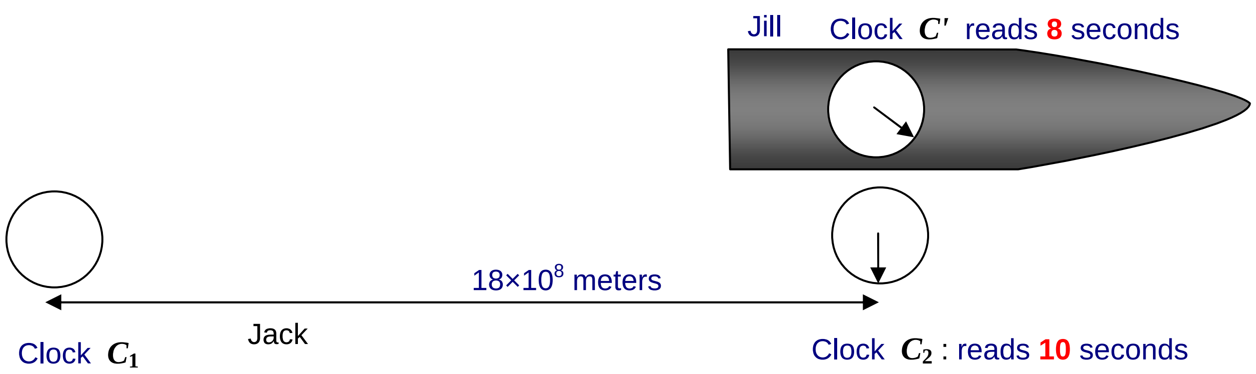
What does clock C' (the clock on the ship) read as it passes C_2?

The time dilation factor

$$\sqrt{1-\left(v^2/c^2\right)} = 4/5$$

so C', Jill's clock, will read 8 seconds.

Thus if both Jack and Jill are at C_2 as Jill and her clock C' pass C_2, both will agree that the clocks look like:



As Jill passes Jack's second clock, both see that his clock reads 10 seconds, hers reads 8 seconds.

How, then, can Jill claim that Jack's clocks C_1, C_2 are the ones that are running slow?

To Jill, C_1, C_2 *are* running slow, but remember they are *not synchronized*. To Jill, C_1 is *behind* C_2 by $Lv/c^2 = (L/c)\times(v/c) = 6\times 0.6 = 3.6$ seconds.

Therefore, Jill will conclude that since C_2 reads 10 seconds as she passes it, at that instant C_1 must be registering 6.4 seconds. Jill's own clock reads 8 seconds at that instant, *so she concludes that C_1 is running slow by the appropriate time dilation factor of 4/5*. This is how the change in synchronization makes it possible for both Jack and Jill to see the other's clocks as running slow.

Of course, Jill's assertion that as she passes Jack's second "ground" clock C_2 the first "ground" clock C_1 must be registering 6.4 seconds is not completely trivial to check! After all, that clock is now a million miles away!

Let us imagine, though, that both observers are equipped with Hubble-style telescopes attached to fast acting cameras, so reading a clock a million miles away is no trick.

To settle the argument, the two of them agree that as she passes the second clock, Jack will be stationed at the second clock, and at the instant of her passing they will both take telephoto digital snapshots of the faraway clock C_1, to see what time it reads.

Jack, of course, knows that C_1 is 6 light seconds away, and is synchronized with C_2 which at that instant is reading 10 seconds, so his snapshot must show C_1 to read 4 seconds. That is, looking at C_1 he sees it as it was six seconds ago.

What does Jill's digital snapshot show? It must be identical—two snapshots taken from the same place at the same time must show the same thing! So, Jill *must also* gets a picture of C_1 reading 4 seconds.

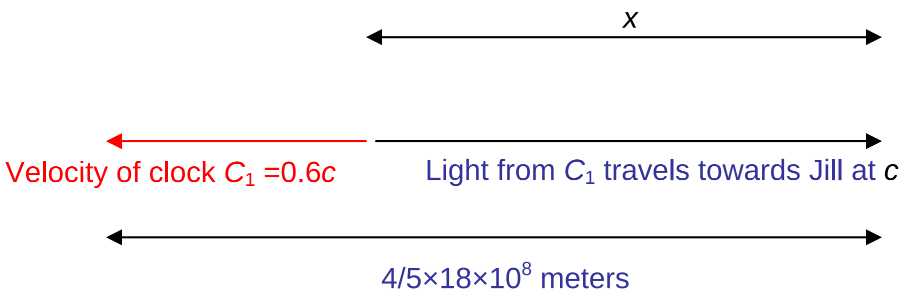
How can she reconcile a picture of the clock reading 4 seconds with her assertion that at the instant she took the photograph the clock was registering 6.4 seconds?

The answer is that she can if she knows her relativity!

First point: length contraction. To Jill, the clock C_1 is actually only 4/5 x 18 x 10^8 meters away (she sees the distance C_1C_2 to be Lorentz contracted!).

Second point: The light didn't even have to go that far! In her frame, the clock C_1 is *moving away*, so the light arriving when she's at C_2 must have left C_1 when it was closer—at distance x in the figure below. The figure shows the light in her frame moving from the clock towards her at speed c, while at the same time the clock itself is moving to the left at $0.6c$.

It might be helpful to imagine yourself in her frame of reference, so you are at rest, and to think of clocks C_1 and C_2 as being at the front end and back end respectively of a train that is going past you at speed $0.6c$. Then, at the moment the back of the train passes you, you take a picture (through your telescope, of course) of the clock at the front of the train. Obviously, the light from the front clock that enters your camera at that instant left the front clock some time ago. During the time that light traveled towards you at speed c, the front of the train itself was going in the opposite direction at speed $0.6c$. But you know the length of the train in your frame is 4/5 x 18 x 10^8 meters, so since at the instant you take the picture the back of the train is passing you, the front of the train must be 4/5 x 18 x 10^8 meters away. Now that distance, 4/5 x 18 x 10^8, is the sum of the distance the light entering your camera traveled plus the distance the train traveled in the same time, that is, (1 + 0.6)/1 times the distance the light traveled.



As Jill passes C_2, she photographs C_1: at that instant, she knows C_1 is $4/5\times 18\times 10^8$ meters away in her frame, but the light reaching her camera at that moment left C_1 when it was at a distance x, not so far away. As the light traveled towards her at speed c, C_1 was moving away at a speed of 0.6c, so the distance $4/5\times 18\times 10^8$ meters is the *sum* of how far the light traveled towards her and how far the clock traveled away from her, both starting at x.

So the image of the first ground clock she sees and records as she passes the second ground clock must have been emitted when the first clock was a distance x from her in her frame, where

$$x(1+3/5) = 4/5\times 18\times 10^8 \text{ meters, so } x = 9\times 10^8 \text{ meters.}$$

Having established that the clock image she is seeing as she takes the photograph left the clock when it was only 9 x 10^8 meters away, that is, 3 light seconds, she concludes that she is observing the first ground clock as it was three seconds ago.

Third point: time dilation. The story so far: she has a photograph of the first ground clock that shows it to be reading 4 seconds. She knows that the light took three seconds to reach her. So, what can she conclude the clock must actually be registering at the instant the photo was taken? If you are tempted to say 7 seconds, you have forgotten that in her frame, the clock is moving at 0.6c and hence *runs slow* by a factor 4/5.

Including the time dilation factor correctly, she concludes that in the 3 seconds that the light from the clock took to reach her, the clock itself will have ticked away 3 × 4/5 seconds, or 2.4 seconds.

Therefore, since the photograph shows the clock to read 4 seconds, and she finds the clock must have run a further 2.4 seconds, she deduces that at the instant she took the photograph the clock must actually have been registering 6.4 seconds, which is what she had claimed all along!

The key point of this lecture is that at first it seems impossible for two observers moving relative to each other to both maintain that the other one's clocks run slow. However, by bringing in the other necessary consequences of the theory of relativity, the Lorentz contraction of lengths, and that clocks synchronized in one frame are out of synchronization in another by a precise amount that follows necessarily from the constancy of the speed of light, the whole picture becomes completely consistent!