Scattering States and Barriers

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Streams of Particles

Our analysis of the time independent Schrödinger equation using the spreadsheet limited us to real values of the wave function $\psi(x)$. This is fine for analyzing bound states in a potential, or standing waves in general, but cannot be used, for example, to represent a stream of electrons being emitted by an electron gun, such as in an old TV tube. The reason is that a real wavefunction $\psi(x)$, in an energetically allowed region, is made up of terms locally like $\cos kx$ and $\sin kx$, multiplied in the full wave function by the time dependent phase factor $e^{-iEt/\hbar}$, giving equal amplitudes of right moving waves $e^{i(px-Et)/\hbar}$ and left moving waves $e^{-i(px+Et)/\hbar}$. So if we are interested in a system in which there are not equal numbers of particles moving to the right and to the left, we must have a wave function such that even the x-dependent part is complex.

A simple example is a stream of particles of energy E moving from the left in one dimension through a region of zero potential, encountering an upward step potential V_0, where $V_0 < E$, at the origin $x = 0$, so that classically the particles would climb the hill and continue to the right. We shall represent the incoming wave function by a plane wave,

$$\psi(x,t) = Ae^{ikx}e^{-iEt/\hbar} \text{ for } x < 0.$$

It proves slightly more convenient to work with wave number k rather than particle momentum $p = \hbar k$ in scattering problems of this type. If we now think of the classical picture of a particle approaching a hill (smoothing off the corners a bit) that it definitely has enough energy to surmount, we would perhaps expect that the wave function continues beyond $x = 0$ in the form

$$\psi(x,t) = Be^{ik_1x}e^{-iEt/\hbar} \text{ for } x > 0,$$

where k_1 corresponds to the slower speed the particle will have after climbing the hill.

Schrödinger's equation requires that the wave function have no discontinuities and no kinks (discontinuities in slope) so the $x < 0$ and $x > 0$ wave functions must match smoothly at the origin. For them to have the same value, we see from above that $A = B$. For them to have the same slope we must have $kA = k_1B$. Unfortunately, the only way to satisfy both these equations with our above wave functions is to take $k = k_1$—which means there is no step potential at all!

Question: what is wrong with the above reasoning?

The answer is that we have been led astray by our mental picture of the particles as little balls rolling along in a potential, with enough energy to get up the hill, etc. Schrödinger's equation is a *wave equation*. Building intuition about solutions should rely on experience with waves. We should be thinking about a light wave going from air into glass, for example. If we do, we realize that at *any* interface *some of the light gets reflected*. This means that our expression for the wave function for $x < 0$ is incomplete, we need to add a *reflected* wave, giving

$$\psi(x,t) = Ae^{ikx}e^{-iEt/\hbar} + Ce^{-ikx}e^{-iEt/\hbar} \quad \text{for } x < 0,$$
$$\psi(x,t) = Be^{ik_1x}e^{-iEt/\hbar} \quad \text{for } x \geq 0.$$

If we now match the wave function and its derivative at the origin, we find

$$A + C = B$$
$$k(A - C) = k_1 B.$$

Recalling that the square of the wave function denotes probability, it is easy to check that the fraction of the wave that is reflected

$$R = \frac{C^2}{A^2} = \left(\frac{k - k_1}{k + k_1}\right)^2.$$

Evidently, the fraction of the wave transmitted

$$T = 1 - R = \frac{4kk_1}{(k + k_1)^2}.$$

Question: isn't the amount transmitted just given by B^2/A^2?

The answer is no. The ratio B^2/A^2 gives the relative probability of finding a particle in some small region in the transmitted stream relative to that in the incoming stream, but the particles in the transmitted stream are moving more slowly, by a factor k_1/k. This means that just comparing the densities of particles in the transmitted and incoming streams is not enough. The physically significant quantity is the *probability current* flowing past a given point, and this is the product of the density *and* the speed. Therefore, the transmission coefficient is B^2k_1/A^2k.

Exercise: prove that even a step *down* gives rise to some reflection.

Barriers

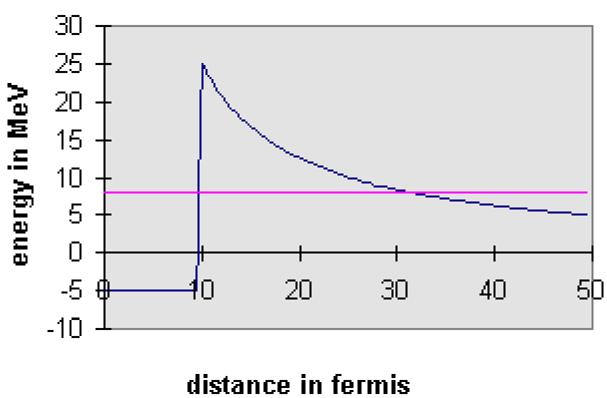
If a plane wave coming in from the left encounters a step at the origin of height $V_0 > E$, the incoming energy, there will be total reflection, but with an exponentially decaying wave penetrating some distance into the step. Suppose now we replace the step with a barrier,

$$V = 0 \quad \text{for } x < 0$$
$$V = V_0 \quad \text{for } 0 < x < L$$
$$V = 0 \quad \text{for } L < x.$$

In this situation, the wave function will still decay exponentially into the barrier (assuming the barrier is thick compared to the exponential decay length), but on reaching the far end at $x = L$, a plane wave solution is again allowed, so there is a nonzero probability of finding the particle beyond the barrier, moving with its original speed. This phenomenon is called *tunneling*, since in the classical picture the particle doesn't have enough energy to get over the top of the barrier.

Alpha decay

A good example of tunneling, and one which helped establish the validity of quantum ideas at the nuclear level, is alpha decay. Certain large unstable nuclei decay radioactively by emitting an alpha-particle, a tightly bound state of two protons and two neutrons. It is thought that alpha-particles may exist, at least as long lived resonances, inside the nucleus. For such a particle, the strong but short ranged nuclear force creates a spherical finite depth well having a steep wall more or less coinciding with the surface of the nucleus. However, we must also include the electrostatic repulsion between the alpha-particle and the rest of the nucleus, a potential $(1/4\pi\varepsilon_0)(Z-2)2e^2/r$ outside the nucleus. This means that, as seen from inside the nucleus, the wall at the surface may not be a step but a barrier, in the sense we used the word above, a step up followed by a slide down the electrostatic curve. Therefore, an alpha-particle bouncing around inside the nucleus may have enough energy to tunnel through to the outside world.



It is evident that the more energetic the alpha-particle is, the thinner the barrier it faces. Since the wave function decays exponentially in the barrier, this can make a huge difference in tunneling rates. It is not difficult to find the energy with which the alpha-particle hits the nuclear wall, because this will be the same energy with which it escapes. Therefore, if we measure the energy of an emitted alpha, since we think we know the shape of the barrier pretty well, we should be able, at least numerically, to predict the tunneling rate. The only other thing we need to know is how many times per second alpha's bounce off the wall. The size of the nucleus is of order 10^{-14} meters, if we assume an alpha moves at, say, 10^{7} meters per second, it will bang into the wall 10^{21} times per second. This is a bit handwaving, but all alpha-radioactive nuclei are pretty much the same size, so perhaps it's safe to assume this will be about the same for all of them. If we do that, we get impressive agreement with experiment over a huge range of lifetimes. polonium^{212} emits alpha's with energy 8.95 MeV, and lasts 3 10^{-7} seconds, thorium^{232} emits 4.05 MeV alpha's, and lasts 1.4 10^{10} years. These can both be understood in terms of essentially the same barrier being tunneled through at the different heights corresponding to the alpha energy. (French, *QM*, page 407).

Exercise: assume the nucleus has a charge of 90e, and a radius of 10^{-14} meters. Estimate the height of the barrier at its maximum, and the width of barrier an alpha must tunnel through for polonium and thorium, discussed above. (Of course, the tunneling rate is not the same as for a rectangular barrier-one must include the variation of the decay length with the changing barrier height. This is the main part of the so-called WKB approximation, see any book on quantum mechanics.)