In this chapter we discuss the basic laws of fluid flow as they apply to life processes at various size scales. For example, fluid dynamics at low Reynolds' number dominates the universe of unicellular organisms: bacteria and protozoa. Fluid flow—both uniform and pulsatile—is the dominant process by which both air and blood circulate in land animals, including humans.

1. Euler's equation

We consider the form of Newton's second law appropriate to a fluid. The mass of a small volume of fluid is

\[ \Delta m = \rho \Delta V ; \]

its acceleration is therefore determined by the forces acting upon it,

\[ \Delta m \frac{dv}{dt} = \rho \Delta V \left( \frac{\partial v}{\partial t} + (v \cdot \nabla)v \right) = \Delta F \, . \]

The forces are generally of two kinds:

1. external, long-range forces—gravity, electromagnetism;
2. internal forces—especially pressure and viscosity.

Thus, e.g., if the fluid is in a gravitational field of local acceleration \( \vec{g} \),

\[ \Delta F = \rho \Delta V \, \vec{g} \, . \]

On the other hand, consider a pressure gradient in—say—the x-direction, as shown to the right. Clearly the net x-component of force is

\[ \Delta F_x = \left( p(x) - p(x+dx) \right) A = - \frac{dp}{dx} \Delta V \, ; \]

In the absence of viscous forces the equation of motion is therefore

\[ \rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right) = - \nabla p + \rho \, \vec{g} \, . \]

This is sometimes known as Euler's equation (L. Euler, 1755).

Example

We first apply Euler's equation to two problems in hydrostatics. In hydrostatics the fluid is not moving, hence \( \vec{v} = 0 \) and we obtain

\[ - \nabla p + \rho \, \vec{g} = 0 \, . \]

Defining \( \vec{g} = -g \, \hat{z} \) we easily see that

\[ p(x,y,z,t) = p(z) \]

so that

\[ \frac{dp}{dz} = -\rho \, g \, . \]

We now have two cases to consider:

1. The fluid is incompressible (\( \rho = 0 \)).
2. The fluid is a perfect gas:

\[ \rho = \frac{\mu}{RT} \rho \, , \]

where \( \mu \) is the gas constant.
where $R = 8.3 \text{ J} \cdot \text{K}^{-1} \cdot \text{gm}^{-1} \cdot \text{mol}^{-1}$ is the gas constant, $T$ the absolute temperature, and $\mu$ the gram-molecular mass.

In Case 1 we may easily integrate the equation to get

$$p(z) = p(0) - \rho g z = p_0 \left(1 - \frac{z}{z_0}\right)$$

where, if $p_0$ is 1 atmosphere ($= 0.76 \text{ m} \cdot \rho_{\text{Hg}}$) and the fluid is water, the height $z_0$ in which the pressure changes by 1 atm is

$$z_0 = \frac{\rho_{\text{Hg}}}{\rho_{\text{H}_2\text{O}}} \times 0.76 \text{ m} = 10.3 \text{ m} \approx 34 \text{ ft}.$$  

In Case 2, we may write

$$\frac{1}{\rho} \frac{dp}{dz} = -\frac{\mu g}{RT}$$

or

$$p(z) = p_0 e^{-\mu g z/RT}.$$ 

The scale height of the atmosphere is then

$$z_0 = \frac{8.3 \times 273}{0.029 \times 9.8} = 8 \text{ km}.$$ 

End of example

2. **Conservation of fluid**

A volume $dV = dx \, dy \, dz$ of fluid contains

$$dN = n \, dV$$

molecules (where $n$ is the number density) and

$$dm = \rho \, dV$$

mass (where $\rho$ is the mass density). If a fluid is flowing with a velocity $v_x$ in the $x$-direction (say), then in time $dt$ we expect

$$dN = n v_x \, dt \, dy \, dz$$

particles to be transported across an imaginary surface of area $dA = dy \, dz$. This leads to the concept of the flux vector

$$\vec{j} = n \vec{v}$$

that describes the transport of particles across the surface. The dimensionality of $\vec{j}$ is $\text{L}^{-2} \cdot \text{T}^{-1}$, that is number per unit area per unit time. Correspondingly we can define the flux vector of mass transport (dimensionality $\text{M} \cdot \text{L}^{-2} \cdot \text{T}^{-1}$)

$$\vec{J} = \rho \vec{v}.$$ 

Now imagine a box with edges $dx$, $dy$ and $dz$ in the respective directions. We erect a vector of unit length pointing outward from each face. The surface integral of any vector over the surface of this box is defined as the sum, over all six faces, of the component of the vector along the outward normal. Thus for the flux $\vec{j}$ we have

$$\int \int \int_{\text{box}} \vec{j} \cdot d\vec{S} = \left[-j_x(x) + j_x(x + dx)\right] dy \, dz$$

$$+ \left[-j_y(y) + j_y(y + dy)\right] dx \, dz$$

$$+ \left[-j_z(z) + j_z(z + dz)\right] dx \, dy.$$ 

This represents the rate at which the particles flow out of the box, that is,

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1. The notation $j_x(x)$ stands for $j_x(x,y,z,t)$; similarly, $j_x(x+ dx)$ means $j_x(x+ dx,y,z,t)$. 

\[ \int \mathbf{j} \cdot d\mathbf{S} = -\frac{d}{dt} (n \, dx \, dy \, dz) = -\frac{\partial n}{\partial t} \, dV. \]

If we then compare both sides (by expanding the small differences \(-j_x(x) + j_x(x + dx)\) to first order) we have

\[ -\frac{\partial n}{\partial t} = \frac{\partial}{\partial x} j_x + \frac{\partial}{\partial y} j_y + \frac{\partial}{\partial z} j_z = \nabla \cdot \mathbf{j}. \]

Note we have defined a new differential operator, the divergence of a vector, as

\[ \text{div} \, \mathbf{j} = \nabla \cdot \mathbf{j} = \frac{\partial}{\partial x} j_x + \frac{\partial}{\partial y} j_y + \frac{\partial}{\partial z} j_z. \]

The equation of continuity, or equation of number conservation is therefore

\[ \frac{\partial n}{\partial t} + \nabla \cdot \mathbf{j} = 0. \]

Suppose the quantity being conserved is something other than particle number— for example, energy, electric charge, mass, or even probability. The conservation laws for such quantities are identical in form, with the number density and number flux replaced with the appropriate density (energy, charge, mass, probability) and the flux vector \(\mathbf{j}\) replaced by the corresponding current density.

3. The Navier-Stokes equation

We now consider viscous fluids. Viscosity arises from the transfer of momentum between a fluid and a solid in relative motion, or between adjacent relatively moving layers of fluid. The derivation of the expression for internal forces resulting from viscosity in the most general case is somewhat advanced and would take us too far afield. Thus we give only the end result in the specialized (and simplified) case of an isotropic, incompressible liquid.

In real life there is no such thing as an incompressible fluid, or solid, for that matter. If one squeezes hard enough, anything is guaranteed to compress. However, when atoms or molecules are touching (or nearly so) as in the case of liquids and solids, the energy needed to decrease even slightly the volume of a given number of molecules is enormous. Hence for practical purposes, over a considerable range of pressure, we can neglect any change in density.

When we assume the density is constant, we see that conservation implies that the divergence of the velocity vector vanishes,

\[ \nabla \cdot \mathbf{v} = 0. \]

The additional force per unit volume of fluid then takes the form \(\eta \nabla^2 \mathbf{v}\) where \(\eta\) is the viscosity. Its dimensionality is \(\text{ML}^{-1}\text{T}^{-1}\). The Euler equation then takes the form (ignoring body forces such as gravitation)

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla p + \eta \nabla^2 \mathbf{v}. \]

This is known as the Navier-Stokes equation.

The physical meaning of viscosity can be understood in terms of the force exerted on a fixed plate by a parallel moving plate, if there is a layer of fluid in between, as shown below:

\[ \frac{F_x}{A_{\text{plate}}} = \eta \frac{\partial v_x}{\partial y}. \]
4. **Reynolds’ number**

Consider an object moving through a fluid. A guide to whether viscosity is important is the ratio of inertial force to viscous force. The inertial force is just, in crude dimensional terms,

\[ ma = \rho L^3 \frac{v^2}{L} = \rho L^2 v^2. \]

The viscous force is

\[ \eta A \frac{\partial u_x}{\partial y} = \eta L^2 \frac{v}{L} = \eta Lv. \]

Their (dimensionless) ratio is therefore

\[ R = \frac{\rho a v}{\eta}, \]

where \( a \) is a typical linear dimension. This ratio is called Reynolds’s number. If \( R \) is large, viscosity is a minor effect; whereas if \( R \) is much smaller than unity, viscosity dominates inertia.

**Example**

Consider various objects moving in water. The viscosity of water is easiest to remember in cgs units: \( \eta = 0.01 \text{ gm/cm/sec} \). Also easy to remember for water is its density: \( \rho = 1 \text{ gm/cm}^3 \).

A typical speed for a rowboat is 1-2 miles per hour, or 50–100 cm/sec. A rowboat is about 3 m long and 1 m wide, hence

\[ R = 1 \times (100 - 300) \times (50 - 100)/0.01 \]

\[ = (0.5 - 3.0) \times 10^6. \]

That is, Reynolds’s number is enormously larger than 1 so viscosity is irrelevant for rowboats, submarines, porpoises, whales etc.

What about an organism the size of a bacterium? Here the length is \( 2 \mu = 2 \times 10^{-6} \text{ m} \), the speed is 30 \( \mu \text{s} \) and \( \rho \) and \( \eta \) are as before. Reynolds’s number is thus \( 6 \times 10^{-5} \) hence a germ’s life is dominated by viscosity. If a spherical bacterium moving at the above speed stops turning its propeller, we find

\[ \frac{1}{v} \frac{dv}{dt} = \frac{dv}{dx} = -\frac{9\eta}{2\rho a^2}, \]

the coasting distance is

\[ x = \frac{2\rho a^2}{9\eta} v_0 \approx 0.1 \text{ Å}. \]

Since the size of an atom is about

\[ 1 \text{ Å} = 10^{-8} \text{ cm}, \]

this is somewhat like our trying to move around by swimming through concrete.

5. **Steady Flow in pipes**

We now use the Navier-Stokes equation to study a case of obvious importance in the physics of the human body: flow of a viscous fluid through a long cylindrical pipe. First we tackle steady (time-independent) flow, then we shall examine pulsatile flow.

Let us take the direction of flow to be the \( z \)-direction. In principle the velocity could have eddies and vortices, even if its average tendency is along the direction of flow. However, the conservation equation

2. In that case, one might well ask, what causes the resistance to propelling a boat through water? It turns out that a displacement hull continuously generates surface waves that carry away energy. A simple calculation shows that the power needed is proportional to \( M v^3/L \) where \( M \) is the ship’s mass, \( L \) its water line length, and \( v \) its speed.

3. Here we have used Stokes’s law, combined with Newton’s second law and the formula for the volume of a sphere. See, e.g., E.M. Purcell, “Life at Low Reynolds Number”, American J. Physics 45 (1977) 3-11.
\[ \nabla \cdot \vec{\nabla} = \frac{\partial}{\partial z} v_z + \frac{1}{r} \frac{\partial}{\partial \phi} v_\phi + \frac{1}{r} \frac{\partial}{\partial r} (rv_r) = 0 \]

in cylindrical coordinates provides some extra information. First of all, the rotational symmetry of the pipe guarantees there is no \( \phi \) dependence. Next, the flow is steady, hence the \( z \)-component of \( \vec{\nabla} \) cannot vary along the pipe. But suppose the radial component, \( v_r \), is non-zero. From fluid conservation we find

\[ rv_r = \text{constant} = A ; \]

clearly, if \( A \neq 0 \), \( v_r \to \infty \) as \( r \to 0 \), that is, the velocity blows up at the center of the pipe. Since that is obviously nonsense, we must have \( A \equiv 0 \) and hence \( v_r \equiv 0 \). There is no radial flow.

Although there might still be a screw-like rotational flow along the pipe, the Navier-Stokes equation tells us that it has no effect in steady flow. That is, the term

\[ (\vec{\nabla} \cdot \vec{\nabla}) \vec{\nabla} \equiv \left( v_r \frac{\partial}{\partial r} + \frac{1}{r} v_\phi \frac{\partial}{\partial \phi} + v_z \frac{\partial}{\partial z} \right) \vec{\nabla} \]

manifestly vanishes.

Before attacking the Navier-Stokes equation per se, we first examine the balance of forces. From the definition of viscosity in terms of velocity gradients the force exerted on an annulus of fluid of length \( L \), lying between radii \( r \) and \( r + dr \), by the fluid outside- and inside it is

\[ \frac{dF_z}{dr} = 2\pi L \eta \left[ \frac{\partial v_z(r+dr)}{\partial r} - \frac{\partial v_z(r)}{\partial r} \right] \]

\[ = 2\pi L \eta \left[ \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) - \frac{\partial p}{\partial z} \right] ; \]

however the force driving the annulus is the pressure differential acting on the ends,

\[ df_z = \Delta p \ 2\pi r \ dr \]

Setting the sum to zero (because the fluid is unaccelerated) we have

\[ \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) = - \frac{\Delta p}{\eta L} r = - \frac{\lambda}{\eta} r . \]

Returning to the Navier-Stokes equation, we note that since the velocity has components only in the \( z \)-direction,

\[ \vec{\nabla} = v_z(r) \hat{z} , \]

\[ \rho \left( \frac{\partial v_z}{\partial t} + v_z \frac{\partial v_z}{\partial z} \right) \equiv 0 = \eta \nabla^2 v_z - \frac{\partial p}{\partial z} . \]

Since there is no radial component of \( \vec{\nabla} \), we see

\[ \frac{\partial p}{\partial r} = 0 , \]

i.e., \( p = p(z) \). But it is also clear that

\[ \nabla^2 v_z = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) \]

is a function of \( r \) alone, as \( \frac{\partial p}{\partial z} \) is a function of \( z \) alone.

Therefore,

\[ \frac{1}{\eta L} \frac{\partial p}{\partial z} = \text{constant} = - \frac{\lambda}{\eta} , \]

and so

\[ \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) = - \left( \frac{\lambda}{\eta} \right) r , \]

\[ r \frac{\partial v_z}{\partial r} = - \left( \frac{\lambda}{2\eta} \right) r^2 + A , \]
\[ v_z = -\left(\frac{\lambda}{4\eta}\right) r^2 + A \log r + B. \]

Manifestly, \( v_z(r = 0) \) is finite, hence \( A = 0 \). On the other hand, the velocity falls to zero at the wall of the pipe, or
\[ v_z = \frac{\lambda}{4\eta} \left( R^2 - r^2 \right). \]

Finally, since
\[ \frac{\partial p}{\partial z} = -\lambda, \]
we can solve to find
\[ p(z) = p_0 - \lambda z = p_0 \left( 1 - \frac{z}{L} \right) + p_L \frac{z}{L}, \]
where \( L \) is the length of the pipe. The result of solving the Navier-Stokes equation is identical to what we got from setting the total force (viscous plus pressure) to zero.

We can now calculate how much mass flows through the pipe per unit time. The flux is \( \rho \vec{v} \); integrating over the cross-section of the pipe we therefore have
\[
\frac{dM}{dt} = 2\pi \int_0^R \rho v_z(r) r \, dr
\]
\[ = \frac{2\pi\lambda\rho}{4\eta} \left( \frac{1}{2} R^4 - \frac{1}{4} R^4 \right)
\]
\[ = \frac{\pi(\rho_0 - p_L) \rho R^4}{8\eta L}. \]
This is known as Poiseuille's Law.

6. **Impedance**

By analogy with Ohm's law we can define an impedance for fluid flow through a pipe. If we take the mass flow to be like electric current and the pressure differential to be like a voltage, then the Poiseuille resistance of a pipe is
\[ Z = \frac{8\eta L}{\pi \rho R^4}. \]
This notion is useful in studying how to optimize the branching of pipes in terms of the total power required.

7. **Pulsatile flow in arteries**

We now consider non-steady flow through a cylindrical tube. Assuming the flow is entirely in the \( z \)-direction and that the velocity depends only on radial distance and time, the Navier-Stokes equation
\[ \rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = -\nabla p + \eta \nabla^2 \vec{v}, \]
reduces to
\[ \frac{\partial \vec{v}}{\partial t} = -\frac{1}{\eta} \frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \vec{v}}{\partial r} \right). \]
Let us suppose \( \frac{\partial p}{\partial z} = \text{Re} \left( -\lambda e^{i\omega t} \right) \) where \( \lambda \) is now complex and the angular frequency \( \omega \) is known. (That is, we analyze the flow for simple harmonic time dependence.) Then we also set \( v(r, t) = u(r) e^{i\omega t} \) and obtain
\[ \frac{1}{r} \frac{d}{dr} \left( r \frac{du}{dr} \right) + \left( \frac{i\rho \omega}{\eta} \right) u = -\frac{\lambda}{\eta}. \]
The solution is
\[ u(r) = \frac{\lambda}{i\rho \omega} + A J_0(\kappa r), \]
where \( J_0 \) is the regular Bessel function of zero'th order and
\[ \kappa = \left( -i\rho \omega / \eta \right)^{1/2}. \]
Since \( u(r=R) = 0 \) we may determine the constant \( A \) to get
\[ u(r) = \frac{\lambda}{i\rho\omega} \left( 1 - \frac{J_0(\kappa r)}{J_0(\kappa R)} \right). \]

Thus
\[ \nu_z(r) = \text{Re} \left( u(r) e^{i\omega t} \right) \]
\[ = \frac{\lambda\omega}{\eta} \left[ \sin \omega t - \frac{M(S)}{M(S)} \sin(\omega t + \theta(S) - \theta(S)) \right] \]
where
\[ s = \kappa r \]
\[ S = \kappa R, \]
and we have written the Kelvin function
\[ J_0(x - i) = M(x) e^{i\theta(x)}. \]

The fluid flow through the pipe can be obtained by integrating \( \rho \vec{v} \) over the area of the pipe, as before. This is facilitated by the relation
\[ \frac{d}{dx} \left[ xJ_1(x) \right] = xJ_0(x). \]

This procedure leads to a complex dynamical impedance for the cylindrical pipe, analogous to the situation in AC circuits.

To determine the flow for the actual time-dependence of the pressure (which is certainly not harmonic, but is periodic) we expand the pressure function in Fourier series. That is, if the pressure looks like the function shown above right, that is, a periodic function
\[ -\nabla p = -\phi(t), \quad \phi(t + T) = \phi(t), \]
we may expand in Fourier series
\[ \phi(t) = \text{Re} \left( \sum_n \lambda_n e^{2\pi i n t / T} \right). \]

Since we have neglected the \( (\vec{v} \cdot \nabla) \vec{v} \) term, the Navier-Stokes equation is linear. Hence the different Fourier components superpose, and we may write
\[ \nu_z(r, t) = \text{Re} \left( \sum_n u_n(r) e^{i\omega_n t} \right) \]
where each Fourier component \( u_n(r) \) of velocity is obtained from the previous formula for \( u(r) \) by replacing \( \lambda \) with \( \lambda_n \) and \( \omega \) with \( \omega_n = \frac{2\pi}{T} \).

The most important thing to note is that the velocity profile across the artery (shown on the next page) is no longer parabolic, but much flatter—thus the rate of transport is proportional to \( R^2 \) rather than \( R^4 \). This has a large effect on optimization of branching for minimum power usage.

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5. Ibid., p. 361.

6. Consult any standard text on applied mathematics, such as M. Mathews and W. Walker, Mathematical Methods of Physics.
Another interesting case is a pipe of rectangular cross section. Here the equation of steady flow is

$$\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} = -\frac{\lambda}{\eta}$$

whose solution may be written (take the origin in the \(x\)-\(y\) plane at a corner of the rectangle)

$$v_z = \sum_{m,n} a_{mn} \sin \frac{m\pi x}{L_x} \sin \frac{n\pi y}{L_y}.$$ 

For this solution the fluid flow rate is

$$\frac{dm}{dt} = \rho \int_0^x dx \int_0^y dy \ v_z(x, y)$$

$$= \rho \frac{4L_x L_y}{\pi^2} \sum_{m,n \text{ odd}} a_{mn}.$$ 

We use the Navier-Stokes equation, together with the orthogonality of the functions \(\sin \left((2n+1)\theta\right)\) to determine \(a_{mn}\):

$$a_{mn} = \frac{16}{\pi^4 mn} \left(\frac{n^2}{L_y^2} + \frac{n^2}{L_x^2}\right)^{-1}.$$ 

Comparing square and circular pipes of equal cross-sectional areas we find the central flow speeds in the ratio

$$\frac{v((1/2, 1/2))_\Box}{v(r = 0)_\bigcirc} \approx 0.93$$

and the rates of mass transport in the ratio

$$\frac{(dm/dt)_\Box}{(dm/dt)_\bigcirc} = \frac{512}{\pi^5} \sum_{m,n \text{ odd}} \frac{1}{(mn)^2(m^2 + n^2)} \approx 0.88.$$ 

It should not surprise us to find that for pipes of equal area, the circular cross section poses the least resistance.

*for advanced students