

Dimensional analysis and scaling laws

1. Dimensional analysis

One of the simplest, yet most powerful, tools in the physicist's bag of tricks is dimensional analysis¹. All quantities of physical interest have dimensions that can be expressed in terms of three fundamental quantities: mass (M), length (L) and time (T).

We express the dimensionality of a quantity by enclosing it in square brackets. Thus, for velocity (change of position divided by change of time) we write

$$[v] = LT^{-1}.$$

Similarly acceleration (change of velocity divided by change of time) has dimensionality

$$[a] = \left[\frac{dv}{dt} \right] = LT^{-2}.$$

From Newton's Second law of Motion,

$$\vec{F} = m\vec{a},$$

we see the dimensions of force are

$$[F] = M \cdot [a] \equiv MLT^{-2};$$

and since work (energy) is force times distance,

$$[E] = [F] \cdot L \equiv ML^2T^{-2}.$$

Dimensional analysis has three important applications:

1. We can avoid algebraic errors by requiring the dimensions of the quantities on two sides of an equation or inequality to be consistent.

2. We can reduce the number of independent parameters in a calculation by re-expressing the problem in terms of relations between dimensionless quantities (pure numbers).

Thus, consider the problem of a physical pendulum, whose equation of motion in the absence of driving torques is

$$I \frac{d^2\theta}{dt^2} + mgl \sin\theta = 0;$$

Let us redefine the time in terms of a dimensionless variable τ and a quantity ω with dimensions of frequency (note θ is already dimensionless because it is defined as a ratio of lengths):

$$t = \frac{\tau}{\omega}.$$

Then the original equation becomes

$$\frac{I\omega^2}{mgl} \frac{d^2\theta}{d\tau^2} + \sin\theta = 0.$$

If we now choose ω so that

$$\frac{I\omega^2}{mgl} = 1$$

our new dimensionless equation is

$$\frac{d^2\theta}{d\tau^2} + \sin\theta = 0.$$

We can make sure we have not made an algebraic error by checking the dimensional consistency of the relation

$$\frac{I\omega^2}{mgl} = 1:$$

Since $[I] = ML^2$ and $[g] = LT^{-2}$ we have

1. In biological and physiological applications dimensional analysis is often called *allometric scaling*.

$$\begin{aligned} \left[\frac{I\omega^2}{mgl} \right] &= M L^2 T^{-2} M^{-1} L^{-1} T^2 L^{-1} \\ &= M^0 L^0 T^0 = \text{constant}. \end{aligned}$$

Hence the expression is dimensionally consistent and contains no obvious algebraic errors².

3. We can “solve” certain physical problems without actually doing the detailed calculations needed for a complete solution.

Thus, suppose we want an expression for the frequency ν of a simple pendulum. The frequency can only depend on the dimensional parameters of the system: the mass m of the bob, the length λ of the string, and the acceleration g of gravity. We suppose the answer to be some pure number (such as π or $\sqrt{2}$) times a product of the parameters each raised to some power:

$$\nu = \text{constant} \times m^\alpha \lambda^\beta g^\gamma.$$

Now require the dimensional consistency of both sides by putting square brackets around all dimensional quantities³:

$$[\nu] = [m]^\alpha [\lambda]^\beta [g]^\gamma$$

which can be rewritten

$$M^0 L^0 T^{-1} = M^\alpha L^\beta \left(\frac{L}{T^2} \right)^\gamma \equiv M^\alpha L^{\beta+\gamma} T^{-2\gamma}.$$

Exponents of like dimensional quantities on both sides of this equation must agree. In general this yields three equations:

$$\begin{aligned} 0 &= \alpha \\ 0 &= \beta + \gamma \\ -1 &= -2\gamma. \end{aligned}$$

The solution is $\alpha = 0$, $\beta = -1/2$, $\gamma = 1/2$. The fact that the exponent of M is zero means that the period does not depend on the mass of the pendulum bob⁴. Hence our formula for the pendulum's frequency becomes

$$\nu = \text{constant} \times \left(\frac{g}{l} \right)^{1/2},$$

a well-known formula. (In fact, for a pendulum started at zero angular velocity from initial angle θ_0 the constant is (for small θ_0)

$$\frac{\sqrt{2}}{4} \left[\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \right]^{-1} \approx \frac{1}{2\pi}$$

but dimensional analysis does not reveal this—to learn its value one must actually solve the differential equation of motion.)

Another point worth making in this context: if there is more than one dimensional quantity of each genre—for example the radius r of the bob, the mass μ of the string—then the above result must be multiplied by an unknown function of dimensionless ratios:

$$\nu = \left(\frac{g}{l} \right)^{1/2} F(\mu/m, r/l, \dots).$$

Surface waves in deep water

We can use dimensional analysis to determine the speed of surface waves on deep water. The quantities in the problem are the wavelength λ , the density ρ of the fluid, and the acceleration of gravity, since the forces are again gravitational. The dimensional equation is

$$\nu = \text{constant} \times \lambda^\alpha \rho^\beta g^\gamma.$$

2. Of course this test will not catch errors of sign or purely numerical factors such as 2 or π .
3. We drop the constant because it has no dimensions.
4. This is characteristic of problems involving the gravitational force.

We are tacitly assuming here that the water depth is so great compared with the wavelength as to be effectively infinite; and that the viscous forces may be ignored. Converting this to dimensional terms we have

$$L^1 T^{-1} M^0 = L^\alpha M^\beta L^{-3\beta} L^\gamma T^{-2\gamma};$$

enforcing dimensional consistency gives, as before, three equations:

$$\begin{aligned} 1 &= \alpha - 3\beta + \gamma \\ -1 &= -2\gamma \\ 0 &= \beta. \end{aligned}$$

The solution is $\gamma = 1/2$, $\alpha = 1/2$, $\beta = 0$. Thus there is no dependence on the density of the fluid; instead the wave speed is

$$v = \text{constant} \times \sqrt{g\lambda}.$$

Again this agrees with the result one obtains by solving the partial differential equations of fluid mechanics, up to the unknown numerical constant.

2. Allometric scaling

Insects have nothing to fear from gravity. No fall in the Earth's gravitational field can kill an insect. A mouse can fall down a 1000 foot deep mine shaft and suffer a minor shock, but no lasting injury, when it hits. A rat, on the other hand, might well die; a man would certainly die; and a horse would *splash*.

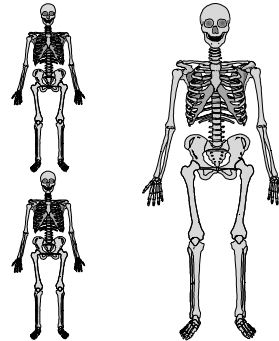
Are mere bugs and mice so much sturdier than larger animals, or is something else taking place here? Galileo Galilei, the great Florentine scientist, was the first to point out that similar objects, no matter what their shapes, can be related by *scaling laws*⁵. These laws explain why an elephant cannot look like an enlarged mouse, why a giant human necessarily has certain problems, and why there are ultimate limits to the sizes of animals, plants and structures.

Strength to weight ratio

To understand this, consider first two cubes. The first is 1 meter on a side, the second 2 meters on a side. The volume of the first is obviously 1 m^3 . That of the second, however, is 8 m^3 . The same holds true of spheres: a sphere of radius 2 m has 8 times the volume of one with a 1 m radius.

In fact, the same is true of the volumes of similar objects of *any* shape: if one is twice as large in any dimension—length, width, thickness—its volume will be 8 times that of the smaller object.

Suppose the two objects are made of the same kind of stuff: for example, gold. Since the density of a



material does not change with size or shape, the object that is twice as tall will have *eight times* the mass. If my height were doubled to 12 feet and all my other dimensions were also doubled, then my weight would be nearly 1 ton. Similarly, were I shrunk to three feet in height, with my other dimensions kept in proportion, I would weigh about 32 pounds.

Exercise

Given the above information, what do I weigh?

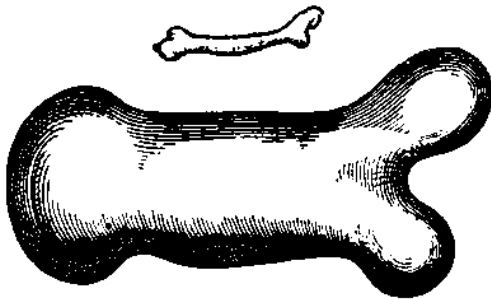
5. Galileo Galilei, *Dialogue Concerning Two New Sciences* (1638).

Structural strength

The strength of tendons and bones increases as their cross-sectional area—that is, roughly as the squares of their linear dimensions. Similarly, the force a muscle can exert in tension also scales as L^2 . Bone, like any other structural material, has strength proportional to its cross-sectional area. If an animal is scaled up in size keeping its bones in proportion, then at twice the linear size the strength, relative to its weight, of its bones is only half as great as those of the smaller animal. More precisely, the strength to weight ratio of geometrically similar animals scales as L^{-1} or $M^{-1/3}$.

This poses a problem for the larger animal—it has to more carefully avoid bangs and bumps than its smaller counterpart, for fear of breaking bones or tearing tendons⁶.

Large animals—rhinoceri, elephants, dinosaurs—solved this evolutionary problem by changing their proportions. Their bones are proportionately much thicker than those of smaller animals. And



Galileo's depiction of the bones of light and heavy animals. (From *Dialogue on Two New Sciences*.)

so are their muscles, whose driving power is proportional to their cross sections. But there is a limit to the thickness that bones or muscles can de-

velop, hence there is a limit to the size of land animals. This seems to have been reached in the largest dinosaurs, which may have weighed 60 tons. Some schools of thought have believed such large sizes were possible only for quasi-aquatic animals, who could have been partly supported by the water of the lakes they waded in.

Suppose the average density of bone is three times that of water (the density of most rocks is about that, e.g.). Let

$$x = \frac{V_{bone}}{V} = \frac{A_{bone} L_{bone}}{V};$$

then the cross-sectional area of the bone (structural strength, that is) must be proportional to the weight of the animal:

$$\sigma_{max} A_{bone} = m g = g [3\rho xV + (1-x)\rho V]$$

where σ_{max} is the maximum stress a bone can bear.

Moreover,

$$L_{bone} = \lambda V^{1/3},$$

since the overall length of the bones in the skeleton must be proportional to the typical linear dimension of the animal. Thus

$$\frac{\sigma_{max} A_{bone} L_{bone}}{\rho g \lambda L V} \stackrel{df}{=} x \frac{L_{max}}{L} = 1 + 2x$$

hence

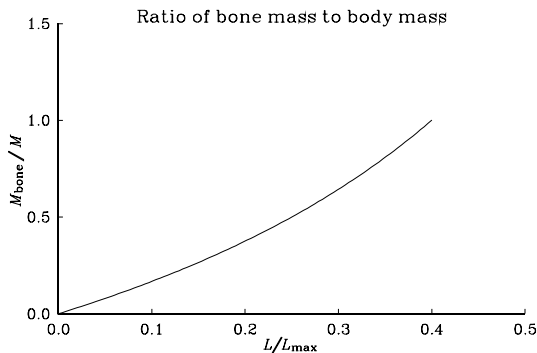
$$\frac{m_{bone}}{m} = 3x = \frac{3}{2} \frac{L}{L_{max}} \left(1 - \frac{L}{L_{max}}\right)^{-1},$$

where L_{max} is a constant with dimensions of length.

For small animals, therefore, the ratio of bone mass to overall mass increases linearly with the linear dimension of the animal. Eventually, however, just to keep the animal from collapsing into a puddle the animal must become all bone—in other words

6. This pertains to football players as well. The greater size of today's players implies a greater rate of injury and a shorter playing career.

it has no room for muscles or organs. The result is shown below.



Whales live in water, hence are not supported by immense bones but rather by buoyancy. Their bones are mostly levers that muscles are attached to. The largest whale ever caught would have weighed about 350-400 tons (length 110 feet). When a whale beaches accidentally or because of disease, it quickly suffocates under its own weight because its supporting structure is inadequate in the absence of buoyancy. The ability of whales to bypass the structural restrictions of self-support was also noted by Galileo⁵.

Terminal velocity

Now, what about a falling mouse, horse or elephant? At low speeds viscosity dominates; but at higher speeds the air's resistance to movement through it is proportional to cross-sectional area. But whereas an elephant-sized mouse would be some 100 times taller than a mouse, and its cross section therefore $100^2 = 10,000$ times greater, its mass would be $100^3 = 10^6$ times as large, hence its ratio of cross-section to mass would be 100 times less.

How does this affect a free fall? The force of gravity is

$$W = -m g,$$

and that of air resistance is

$$F_{Air} = \Gamma A v^2$$

where Γ is a constant proportional to the air density. Dimensional analysis would then say

$$[v] = LT^{-1}M^0 = [g^\alpha m^\beta \Gamma^\gamma A^\delta]$$

or

$$v = \text{constant} \times \left(\frac{mg}{\Gamma A} \right)^{1/2} \propto L^{1/2}.$$

We can reach the same conclusion using Newton's Second Law of motion

$$F_{Tot} = W + F_{Air} = m a$$

or

$$\frac{dv}{dt} - \frac{\Gamma}{m} A v^2 = -g.$$

This equation can be integrated by separation of variables:

$$\frac{dv}{\lambda v^2 - g} = dt$$

where $\lambda = \Gamma A/m$, giving

$$v(t) = - \left[\frac{mg}{\Gamma A} \right]^{1/2} \tanh(t/\tau)$$

where $\tau = \sqrt{g\Gamma A/m}$ is a characteristic time (that could also be found by dimensional analysis). Initially the gravitational force accelerates the object downward with acceleration $\dot{v} = -g$. As it picks up speed, air resistance becomes important, so the net force decreases and the acceleration \dot{v} falls to zero. When this happens, the object remains at a constant speed, the *terminal velocity*. This is shown on the following page.

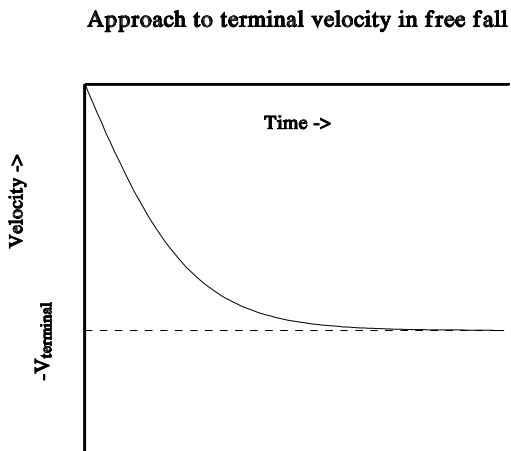
The asymptotic speed of impact increases with the ratio of weight to surface area, which means it increases as the square root of the linear size of the animal:

$$v_{term} \propto L^{1/2}.$$

How high can you jump?

To jump a height h an animal must expend energy

$$E = mgh.$$



To accomplish this it must exert a force F over a distance D so that

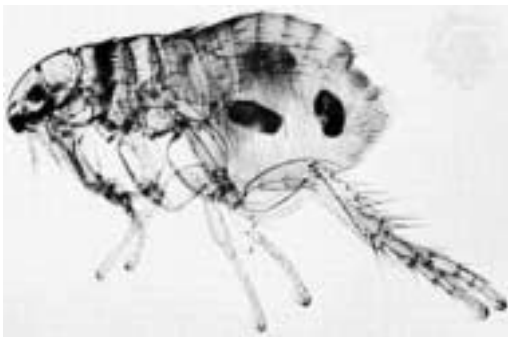
$$F \cdot D \sim mgh.$$

Muscular forces scale as the cross-sectional area of the muscles involved, *i.e.* as L^2 where as usual, L is the linear size of the animal. Obviously D , the distance the muscle contracts, also scales like L .

The mass scales as volume, L^3 , so we find

$$h \sim \frac{L^3}{L^3} \times \text{constant}$$

—the height of the jump is independent of the animal's size! Or to put it another way, scaling suggests that a flea can jump as high as a man, and



*vice versa*⁷. Paradoxical as this may sound, it is indeed correct. A flea can jump about 20 cm straight up, and a human about 60 cm. While the (human) world record in the (running) high jump exceeds 230 cm, such jumps are accomplished by converting forward motion into vertical motion⁸, as well as by adroitly timed movements of arms and legs, that allow the athlete's center of mass to remain well below the bar at all times.

Basal metabolism

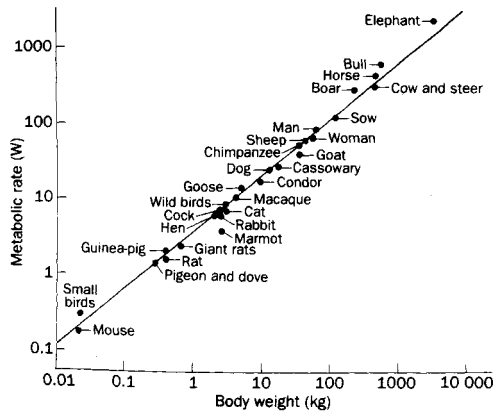
What determines the ultimate length a whale can grow to? Several factors combine to limit its size (which may be the largest possible size for any mammal): the need for oxygen and food increases as the cube of its length, but the generation of energy to sustain life also means the production of waste heat. The surface of a whale must be sleek so it can swim rapidly. This means it can not have projecting radiator fins (other than its swimming apparatus, of course). But then the whale's ability to get rid of its excess body heat, even in very cold water, is limited by the surface area, which increases only as the square of its length.

Basal metabolism of mammals (that is, the minimum rate of energy generation of an organism) has long been known to scale empirically as

$$B = \frac{dQ}{dt} = \text{const.} (Mass)^{3/4}.$$

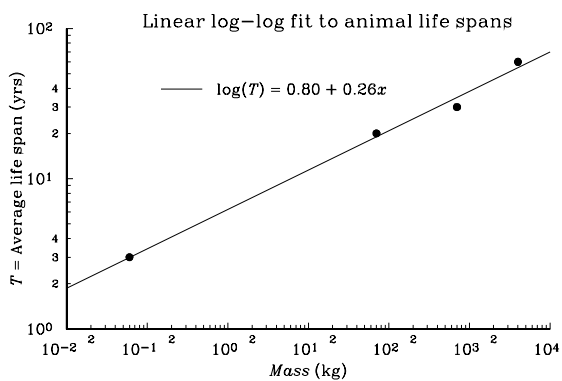
The origin of this relation, graphed on the following page, sometimes called Kleiber's Law, has recently been explained by West, *et al.*⁹ in terms of optimizing the pumping efficiency for fluid flow in the circulatory and pulmonary systems of mammals. They note that the terminus of a capillary or alveolar duct must necessarily be of constant size, independent of animal mass. Since the arterial

7. The first person to analyze this seeming paradox was Giovanni Alfonso Borelli (1608-1679) in his book *De motu animalium*.
8. Indeed, the jumper is using his leg as a sort of pole-vaulting pole.
9. G.B. West, J.H. Brown and B.J. Enquist, *Science* **276** (1997) 122.



network and bronchial systems each have a tree-like structure, and since the sub sections of the tree are self-similar (“fractal”) what determines the size of the entry—the aorta or the trachea, respectively—is the ratio of branch diameter to branch length, and the fact that the branches are (almost) always bifurcations¹⁰. When this ratio is chosen to minimize resistance to flow, hence pumping power, Klieber’s Law emerges.

The largest whales are certainly at the ragged edge, maintaining a precarious balance between energy production and heat dissipation. When a whale dies (by being killed by hunters, *e.g.*) and its heart stops circulating the blood (which acts like the



coolant in a radiator), its flesh actually *cooks* within its jacket of blubber because the residual metabolic heat production has no way to escape. The temperature rises, therefore (the onset of decay from bacterial action accelerates this process). Some of the old-time whalers apparently enjoyed whale meat “cooked” in this fashion¹¹!

Lifetimes

Suppose the heart- and blood volumes both scale proportionally with the volume of the animal. The rate of circulation, $\frac{dV}{dt}$, must be proportional to the basal metabolism, so we can say the frequency at which the heart beats is

$$f \cdot V_{heart} = \frac{dV_{blood}}{dt} = c \cdot M^{3/4}.$$

But since the mass is proportional to the volume we have a scaling law

$$f \sim M^{-1/4}.$$

If we suppose that each animal’s heart beats a fixed number of times during its lifetime,

$$N = f \cdot \tau = \text{constant},$$

we see that lifetimes should scale as

$$\tau \sim M^{1/4}.$$

As the graph to the left shows, this “law” is satisfied very well empirically.

Brain size

Interestingly, the brain mass also scales as body mass to the $3/4$ power,

$$M_{brain} \sim M^{3/4}$$

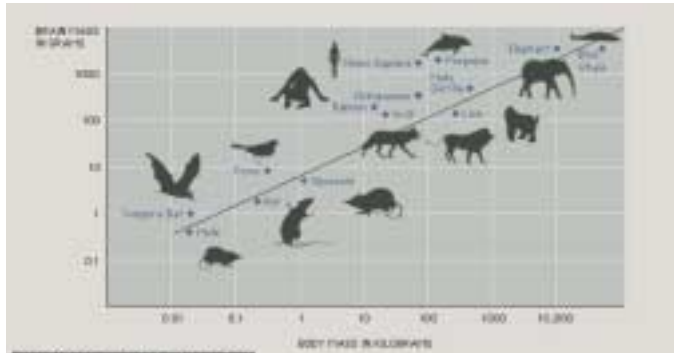
(see figure on following page).

The data fall on a band with slope $3/4$ on a log-log graph, with primates occupying the upper edge of

10. The physics of fluids determines that bifurcation is better than, say, trifurcation.

11. Herman Melville, *Moby Dick*.

the band, and modern humans represented by a point well above the band. That is, *Homo sapiens* has a substantially larger ratio of brain to body mass than any other species.



Two interesting questions arise from this empirical relation. First, why—like Kleiber’s Law for basal metabolism—is the power $3/4$? And second, why is the human brain so much larger (1.5–2× larger) than the empirical scaling would predict?

Much of the brain’s computational power is devoted to muscular control. An elephant’s trunk, e.g. has 6 major muscle groups divided into about 10^5 individually controllable muscle units. Its brain weighs 3.6–5.4 kg. For comparison, there are only 639 muscles in the human body, and human brains weigh about 1.3 kg.

Kleiber’s Law for basal metabolism followed from the self-similar character of arterial or bronchial networks⁹. Similarly, the density of neuromuscular junctions is presumably constant; however to optimize the power demand of the brain segment that controls the musculoskeletal system, requires the number of neurons per unit of muscle to diminish

with body size. This optimization can be effected *via* bifurcation of axons—that is, the neural network forms a self-similar bifurcating tree like the arterial network, where the constraints on transmission of nerve impulses determine the scale of bifurcations.

Among mammals, apes, elephants and whales have brains larger than the $M^{3/4}$ fit would predict. When we examine the elephant brain we see it has massive temporal lobes and huge sections devoted to controlling the trunk muscles. The temporal lobes provide the proverbial “elephant’s memory”¹²; and the sections controlling the trunk musculature have to be enormous because this is the animal’s most vital organ¹³. Not much brain is left over for abstract reasoning, and elephants do not seem to have such ability. The dolphin and orca (killer whale) are fast swimmers, and also possess extremely sophisticated sonar. They achieve their speed by constantly adjusting thousands of subcutaneous muscles that cancel out turbulence—and thereby greatly reduce drag. The dolphin brain therefore devotes much of its computational power to the sensory and feedback control elements of the drag-reducing and sonar systems. Though their brains are moderately larger than humans’, dolphins’ intelligence ranks somewhere between wolves and chimpanzees¹⁴.

Since the central nervous system is a great consumer of energy¹⁵, we conclude that evolutionary parsimony gives animals the smallest possible brains consistent with species survival. This accounts for the brains (or lack thereof) of the rest

12. ...needed to recall the locations of oases and water holes along the migration paths that elephant herds must follow.
13. Elephants with injured trunks soon die.
14. I feel impelled to add, however, that the killer whale has always impressed me as having human-level intelligence. Studies of their behavior in the wild reveal a level of planning and forethought that are unmatched by any animal other than human beings.
15. Neurons require operating power 10× the average of other cells; moreover, the energy needed to create a zygote’s central nervous system places severe demands on the mother during gestation.

of the animal kingdom, but does not really explain why humans took the odd evolutionary path that led to intelligence. Perhaps that path was mandated by the development of the human hand, with all that implies.

