Assignment 10

1. Radiation and scattering: why is the sky blue? As one application of the Larmor radiation formula we will study the excitation of an atom by an incident plane EM wave. The interaction of the electric field of the incident plane wave with the electrons and the nucleus causes them to oscillate at the same frequency as the wave, resulting in an oscillating electric dipole (the magnetic forces are entirely negligible except for very intense incident waves.) The accelerations of the electrons are large compared to the accelerations of the nucleus, due to the large mass ratio. The oscillating dipole then causes the atom to radiate electric dipole radiation, as discussed in class.

(a) Consider a simple model in which one electron is bound harmonically to the nucleus (i.e., the electron is subject to a restoring force, $-m\omega_0^2 z$, with m the electron mass, ω_0 a natural atomic frequency, and z the displacement from the origin, taken to be the equilibrium position of the electron). The electric field of the incident plane wave at the atom has the form $E_z(t) = E_0 \cos(\omega t)$ (we can manage well without complex notation in this problem). Assuming that $\omega \ll \omega_0$, show that the electron moves in phase with the electric field with an acceleration

$$\frac{d^2 z}{dt^2} = \frac{eE_0\omega^2}{m\omega_0^2} \cos(\omega t). \tag{1}$$

(b) Use the Larmor radiation formula to show that the time-averaged power radiated by the charge is

$$P_{av} = \frac{1}{4\pi\epsilon_0} \frac{e^4 E_0^2}{3m^2c^3} \left(\frac{\omega}{\omega_0}\right)^4. \tag{2}$$

Express this in terms of the wavelength λ of the incident wave to show that $P_{av} \propto \lambda^{-4}$, which is a famous law derived by Lord Rayleigh. This results shows that short wavelength radiation is scattered more effectively by atoms than long wavelength radiation. This result is valid for $\lambda \gg a$, with a the characteristic size of the atom.

(c) Use the above result to explain (i) why the sky is blue, (ii) why sunsets are red, and (iii) why it is easier to get sunburned at midday.

(d) finally, think a bit about the polarization of the scattered radiation. Suppose you take some Polaroid sunglasses and look northward as the sun sets in the west. If you rotate the sunglasses you will notice marked intensity variations (try it). Why? Please be as specific as possible. figure 4.3(c) in *Melissinos* may help. See also the lecture notes.

(a) We start with the very simple picture of an atom in the gas as an electron bound to a much heavier nucleus by a spring — this is what is meant by "bound harmonically". The equilibrium position of the spring has the electron directly sitting on the nucleus. (This is a very naive model; more accurately the electron makes a transition between orbits and the "position" corresponds to the center of the orbit, which is stretched by the applied field.) But when the electron is at other positions we have the negatively charged electron separated from the positively charged nucleus by some distance, that is, we have an electric dipole ($p = qd$ where d is the distance separating the two charges $\pm q$ and the dipole points along that length from the negative to the positive charge). So if we were to apply a constant field E_z to the simple harmonic atom, we'd get a displacement $z = -eE/k$, where k is the spring constant, and the resulting dipole would be $\mathbf{p} = \hat{\mathbf{z}}\, e^2 E/k$. Next we consider the situation in which the applied field is sinusoidal as it would be if it is the electric field associated with electromagnetic radiation (i.e. light).

$$\begin{aligned} F &= ma \\ -eE_0\cos(\omega t) - kz &= m\frac{d^2z}{dt^2}, \end{aligned}$$

where we neglect damping. (*Melissinos* includes it.) The steady-state solution to this equation is

$$z = \frac{-eE_0\cos(\omega t)}{m(k/m - \omega^2)}.$$

Compare this to the solution of the LRC circuit in Assignment 8. The natural frequency is given by $\omega_0^2 = k/m$; in the limit $\omega \ll \omega_0$ we have

$$z = \frac{-eE_0\cos(\omega t)}{m\omega_0^2}.$$

And the corresponding acceleration is

$$\frac{d^2z}{dt^2} = \frac{eE_0\omega^2\cos(\omega t)}{m\omega_0^2}.$$

(b) Recall that **accelerating charges radiate**. The Larmor radiation formula gives the power radiated by an accelarting charge as

$$P = \frac{2}{3}\,\frac{e^2}{4\pi\epsilon_0}\,\frac{\dot{v}^2}{c^3},$$

(*Melissinos'* eq. (4.13)). Substituting the expression for acceleration found above, we get

$$P = \frac{e^4E_0^2\omega^4\cos^2(\omega t)}{6\pi\epsilon_0 m^2\omega_0^4 c^3}.$$

finally expressing this result in terms of wavelength ($\omega = c/\lambda$) results in

$$\langle P\rangle_t = \frac{e^4E_0^2c}{12\pi\epsilon_0 m^2\omega_0^2\lambda^4},$$

where $\langle\rangle_t$ stands for the time average. Note that this is the formula found by Lord Rayleigh.

(c) (i) The above result on radiation due to "scattering" implies that blue light is scattered more than red light. Thus the sky is blue because the light of the sky is scattered light (we are not looking directly toward the source of light). Violet light should be scattered even more; so why isn't the sky violet?

(ii) When viewing sunsets we look more directly at the source of radiation. The blue light is scattered away, leaving the red light.

(iii) In the simplest scenario we are concerned with direct, ultraviolet light, because this light causes the greatest damage to the human skin. (So we will not worry about the scattered or reflected light, and we will not worry about the relative amounts of other frequencies in the direct light.) When the sun is low in the sky, the direct unltraviolet light travels through more atmosphere and therefore more of it is scattered.

(d) Light is a *transverse* wave which means that as it travels, say in the y direction of the figure above, the electric field lies in the xz plane. Therefore, the dipoles excited in the atmosphere also lie in the xz plane. An oscillating dipole does not radiate along its axis (see fig. 4.3 in *Melissinos)*, so dipoles oscillating in the z direction do not send light to the eye. So only the component of the dipole in the x direction sends radiation to the eye (along the z direction). If the viewer orients his or her polaroid sunglasses to admit radiation with the electric field pointing in the x direction, there is scattered light with that polarization. On the other hand, if he or she orients the sunglasses in the y direction, the amount of light should be noticeably reduced.

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2. The plasma frequency and the ionosphere.

(a) Note that eq. (4.34) in *Melissinos* becomes equivalent to eq. (4.42) or (4.44) in the appropriate limit. What is this limit and what does it mean physically?

(b) The plasma frequency for the ionosphere is given on page 135 by *Melissinos*. Electrons in a metal behave much like those in the ionosphere, but their density is much higher. Estimate the plasma frequency of copper and explain why many metals become transparent in the ultraviolet.

(a) In the earlier formula in *Melissinos* and in problem 1 in this set, an atom or molecule was modeled as an harmonic oscillator (damped in *Melissinos*). In a plasma the electron is not bound to the nucleus. Hence the appropriate limit is $k \to 0$, i.e. no spring coupling the electron and nucleus (in *Melissinos*'s case, the damping γ must also be taken to zero). The index of refraction n, Eq. (4.34), is expressed in terms of the natural frequency ($\omega_0^2 = k/m$), so $\omega_0 \to 0$.

$$\lim_{\omega_0 \to 0} \lim_{\gamma \to 0} \left[1 + \frac{Ne^2}{m\epsilon_0} \frac{1}{(\omega_0^2 - \omega^2) - i2\gamma\omega} \right]^{1/2} = \left[1 - \frac{Ne^2}{m\epsilon_0 \omega^2} \right]^{1/2},$$

which is the same as the formula for the index of refraction in the ionosphere, eq. (4.44) in *Melissinos*, if

$$\omega_p^2 = \frac{Ne^2}{m\epsilon_0}.$$

This quantity ω_p is called the plasma frequency. N is the number density of electrons, and m the mass of the electron.

(b) Metals are like the plasma of the ionosphere in that the outer electrons are not bound to nuclei.

The carrier density N of copper is $8.48 \times 10^{28}\ \mathrm{m}^{-3}$; the permittivity of free space ϵ_0 is $8.85 \times 10^{-12}\ \mathrm{C}^2/\mathrm{N} \cdot \mathrm{m}^2$; the mass of the electron m_e is 9.109×10^{-31} kg; and the charge

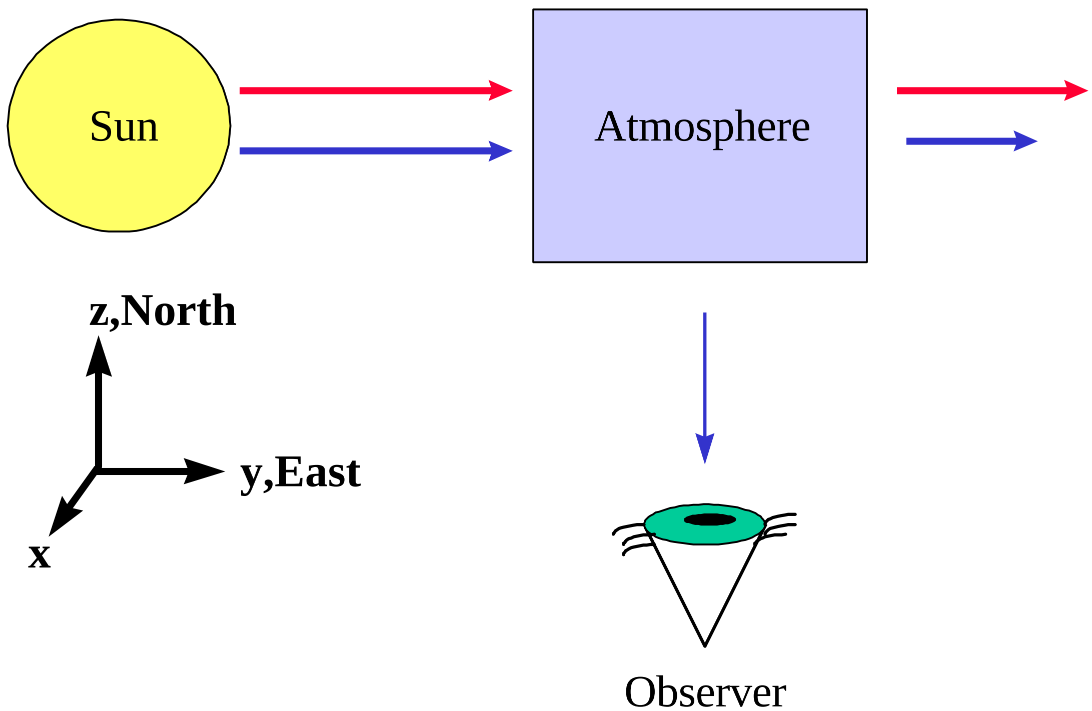

Figure 1:

of the electron e is 1.602×10^{-19} C. Thus the plasma frequency is

$$\omega_p = \left[\frac{(8.48 \times 10^{28}\ \mathrm{m}^{-3})(1.602 \times 10^{-19}\ \mathrm{C})^2}{(9.109 \times 10^{-31}\ \mathrm{kg})(8.85 \times 10^{-12}\ \mathrm{C}^2/\mathrm{N} \cdot \mathrm{m}^2)} \right]^{1/2}$$

or $\omega_p = 1.64 \times 10^{16}$ Hz. When $\omega < \omega_p$, the index of refraction is imaginary; and when $\omega > \omega_p$ the index of refraction is real.

What does it mean to have an imaginary index of refraction? A simple plane wave, be it in vacuum or in a material, has the form

$$\exp\{ikx - i\omega t\}.$$

When the wave hits an interface between two materials, the incident, reflected and transmitted waves must all have the same frequency. So all of the waves will share the $\exp\{-i\omega t\}$ dependence. The wavenumber and angular frequency are related through $\omega = vk$, where v is the speed of the wave, so that the form of the wave can be written as

$$\exp\{i\omega x/v - i\omega t\}.$$

Furthermore, the speed of the wave is related to the index of refraction by $v = c/n$, thus we get

$$\exp\{in\omega x/c - i\omega t\}.$$

If n is real this is an oscillating function of x; if n is imaginary it is an exponential function of x

$$\exp\{-|n|\omega x/c - i\omega t\},$$

meaning that the wave decays in the material rather than propagates through it — no transmitted wave. Thus for $\omega < \omega_p$ the index of refraction is imaginary and the transmitted wave does not propagate through the medium; whereas for $\omega > \omega_p$ the index of refraction is real and the transmitted wave does propagate. For a metal such as copper ω_p is in the ultraviolet part of the spectrum.

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3. Physics vocabulary. The following terms (which are used by *Melissinos*) have specific meanings in the physical sciences, although they may seem equivalent or vague to the uninitiated.. Give a short definition and an example of each.

(a) Diffusion
(b) Dispersion
(c) Diffraction
(d) Refraction
(e) Birefringence
(f) Scattering

The terms above apply to a broad class of phenomena and in some situations they overlap. All of the terms above can be used to refer to light, and that is the context in which we discuss them below. However, the terms can be used to describe other waves, and in the cases of scattering and diffusion are not restricted to waves. We will take the terms out of order as it suits the discussion.

(c) Consider a plane wave of light that is partly obstructed, as when it passes through a slit or aperture: according to the Huygens' principle each point along the unobstructed wavefront can be considered a point source of light. One outcome is that the wavefront is no longer flat planes but bends. Another outcome is that these point sources can interfere giving rise to a diffraction pattern. This combination of bending and interfering of an obstructed wave is called **diffraction**. (Strictly speaking, we can take the same point of view of the plane wave prior to the obstruction, but the sum of all the interfering points sources along a plane is just another plane.) While diffraction can occur in media, and in fact be used as a tool to characterize materials; except for the obstruction, a medium is not necessary to have diffraction. Diffraction occurs in and sets the limits of resolution for microscopes, telescopes and so forth.

(d) The speed of light varies from material to material and is characterized by the index of refraction $n_i = c/v_i$ where v_i is the speed of light in the material, and c is the speed of light in a vacuum. **Refraction** refers to the bending of a ray of light when it encounters an interface separating two materials with different indexes of refraction. The amount of bending is given by Snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

where the angles θ_1 and θ_2 are measured from the ray to the normal to the plane separating the materials. An example is the bending of light that occurs as a ray of light passes through a lens.

(b) If the index of refraction varies with the wavelength of light, the amount a ray bends when it encounters an interface between two materials also varies with wavelength. White light, which is composed of a continuum of frequencies from the visible spectrum, gets separated into its constituent wavelengths since each bends a different amount and travels a different path. This phenomenon is called **dispersion**. Light passing through a prism is an example of dispersion. Recall that a pulse of light is composed of a number of wavelengths — if it is moving through a *dispersive* medium, the different components travel at different speeds and thus the pulse changes it shape.

(e) In a **birefringent** material the index of refraction depends on the polarization of light. That is, the speed of electromagnetic radiation depends on the orientation of the electric field. These materials are, of course, anisotropic; calcium carbonate (calcite) is the usual example. Here the amount of bending depends on polarization. Typically the electric field has a component along each axis; these components get split apart, much as the colors did in dispersion. Birefringent materials can be used to alter the polarization of light from one direction to another (a half-wave plate) or from linearly to circularly polarized (a quarter-wave plate).

(f) In refraction we view the medium as homogeneous with the interesting phenomena occurring at the interface between two different media. In **scattering** we view the medium as inhomogeneous, particles well separated from one another, or perhaps larger particles amongst many more smaller and closely packed particles. Scattering is then the interaction of a ray of light with the particle (absorption) which then serves as a new source of radiation (reradiation). We can see a laser beam from the side because some of the light is scattered from dust particles.

(a) If light gets scattered many times, it performs a *random walk* through the medium, that is, its direction from scatterer to scatterer is random. The motion of light then is something like a *Brownian particle* and is said to be undergoing **diffusion**. It is also called *multi-scattering*. Light coming from clouds or milk is an example.

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4. Light sources. Briefly describe the characteristics of

(a) Light emitted from a sodium lamp (looks yellow)

(b) Natural light from the sun

(c) Synchrotron radiation

(d) Laser light

In each case, is the light broad-spectrum or narrow spectrum, polarized (at least partly) or unpolarized, coherent (at least partly) or incoherent?

(a) In a sodium vapor lamp the radiation comes from spontaneous emission — "emission" refers to photons given off when the atoms change states; "spontaneous" refers to the fact that one electronic transition does not induce others. Thus the transitions and the photons given off are uncorrelated. In a wave description, their phases are random, and the wave is unpolarized. The radiation from sodium is dominated by a couple of nearby emission lines which correspond to yellow light, and hence, the spectrum is narrow.

(b) Natural sunlight is for the most part thermal or blackbody radiation, radiation given off simply because the sun has a temperature. The emission is totally uncorrelated, random in polarization and phase, and broad in spectrum.

(c) Synchrotron radiation is the light given off by electrons (or other charged particles) as they are forced to move around in large circular paths at velocities approaching the speed of light in particle accelerators, and also in various astronomical situations. See the file syncrad.html in the notes. Each particle emits a coherent, polarized pulse, but there is no coherence between pulses emitted by different particles.

(d) Laser light is due to "stimulated emission" — electronic transitions in which one emission induces further transitions leading to a correlated or phase coherent radiation. The transitions are all the same so the spectrum is narrow, and can be made very narrow. Usually the laser is designed to emit light that is highly polarized.

	spectrum	polarization	coherence
sodium lamp	narrow	unpolarized	incoherent
sun light	broad	unpolarized	incoherent
synchrotron rad.	broad	polarized	partly coherent
laser light	very narrow	polarized	coherent