Assignment 11

1. Solar temperatures. Here we will try to estimate some relevant temperatures in the sun. Assume you start with an initially diffuse cloud of hydrogen and helium atoms (initially at rest), which subsequently collapse under its gravitational attraction.

(a) Use dimensional analysis to show that the total energy released in the gravitational collapse is about

$$E_\odot = \frac{G\,M_\odot^2}{R_\odot} \tag{1}$$

Calculate this energy.

(b) How does this energy compare to the total energy that could be released by (hypothetical) chemical reactions in the sun? To the total energy that could be released by converting all the hydrogen into helium? (You can assume that the cloud has the "primitive" abundance of helium—it does not make much difference as long as you have mostly hydrogen).

(c) Assuming that all of the energy (1) is converted into heat (a rather dubious assumption), estimate the interior temperature of the sun. Compare with the accepted value of the temperature at the center of the sun (look it up).

(d) The intensity of solar radiation has a peak at a wavelength 490 nm. What is the surface temperature of the sun? As you go from the surface of the sun toward the center, what is the approximate temperature gradient dT/dx? How can this temperature gradient be maintained?

(a) The energy it takes to bring two point masses m_1 and m_2 from infinitely far away to within a distance r is

$$E = -G\frac{m_1 m_2}{r},$$

where $G = 6.6726 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$, this is the gravitational potential energy. Dimensional analysis suggests that the gravitational energy associated with collapsing a diffuse cloud to an object the size and mass of the sun is

$$E = -G\frac{M_\odot^2}{R_\odot}$$

(The negative sign indicates that it does not require energy, instead energy is gained.) For the sun which has a mass $M_\odot = 1.99 \times 10^{30}$ kg and radius $R_\odot = 6.96 \times 10^8$ m, we get

$$E = -3.8 \times 10^{41}\ \text{Joules}.$$

A digression. If we assume the density is constant, we can calculate the numerical coefficient. Let us take m_1 to be a sphere of radius r and mass density ρ (it looks like a point mass so long as we stay outside its radius), m_1 is

$$m_1 = \rho \frac{4\pi r^3}{3}.$$

Let us take m_2 to be a spherical shell which after it is brought in from infinity has a radius r, a thickness dr and the same mass density ρ, then m_2 is

$$m_2 = \rho\, 4\pi r^2 dr.$$

The change in energy required to bring in the shell is thus

$$dE = -G \frac{\rho \frac{4\pi r^3}{3}\ \rho 4\pi r^2 dr}{r}.$$

So the energy to make a sphere of radius R is

$$\begin{aligned} E(R) &= -\frac{16\pi^2 G\rho^2}{3} \int_0^R r^4 dr \\ &= -\frac{16\pi^2 G\rho^2}{15} R^5 \end{aligned}$$

Now expressing ρ in terms of the total mass M

$$\rho = \frac{M}{\frac{4\pi R^3}{3}}$$

and substituting we find

$$E(R) = -\frac{3GM^2}{5R} = -2.28 \times 10^{41} \text{ Joules},$$

if the density were constant.

(b) The proton-proton cycle essentially turns four protons and two electrons into an alpha particle and gives off 26.7 MeV in the process. Assuming the sun starts off entirely as protons and electrons and is converted entirely into helium (alpha particles), then the total energy would be

$$\frac{1.99 \times 10^{30} \text{ kg}}{1.67 \times 10^{-27} \text{ kg}} \times \frac{2.67 \times 10^{7} \text{ eV}}{4} \times \frac{1.60 \times 10^{-19}\text{Joules}}{\text{eV}} = 1.3 \times 10^{45} \text{ Joules},$$

which is the number of protons divided by four multiplied by the energy released in a proton-proton cycle. A hypothetical chemical reaction would have an energy 10^5 to 10^6 times smaller. Thus, the hypothetical chemical energy is smaller than the gravitational energy which is smaller than the fusion energy.

(c) Assuming all of the energy calculated in part (a) is converted into heat (thermal kinetic energy), we get

$$\langle K.E. \rangle_{total} = -E_{dim}.$$

(In a similar problem given by Kittel and Kroemer, one is told to assume that *half* of the gravitational potential energy is converted into thermal kinetic energy in accordance with the *virial theorem*, but we are only concerned with order of magnitude here.) Let us suppose this is the average kinetic energy everywhere in the entire sun. Then the average kinetic energy per particle

$$\langle K.E.\rangle_{particle} = \frac{\langle K.E.\rangle_{total}}{N},$$

where N is the total number of particles. Again assuming the sun is composed purely of hydrogen, let N be the number of hydrogen atoms

$$N = \frac{M_{\odot}}{m_p} = \frac{1.99 \times 10^{30}\ \mathrm{kg}}{1.672 \times 10^{-27}\ \mathrm{kg}} = 1.19 \times 10^{57},$$

so then

$$\langle K.E.\rangle_{particle} = \frac{3.8 \times 10^{41}\ \mathrm{Joules}}{1.19 \times 10^{57}} = 3.2 \times 10^{-16}\ \mathrm{Joules}.$$

If we take the equipartition theorem for a monatomic gas then

$$\langle K.E.\rangle_{particle} = \frac{3}{2} k_B T,$$

(where we have assumed the temperature is the same throughout, which is not consistent with what we find later). But anyway this leads to

$$T = \frac{2 \cdot 3.2 \times 10^{-16}\ \mathrm{Joules}}{3 \cdot 1.38 \times 10^{-23}\ \mathrm{J/K}} = 1.5 \times 10^{7}\ \mathrm{K}.$$

Melissinos (p. 193) lists the temperature at the interior of the sun as $T_{\odot}^{int} \sim 1.6 \times 10^{7}$ K.

(d) The electromagnetic radiation from the sun is determined by its surface temperature assuming it radiate as a black-body. *Wien's displacement law* says that $\lambda_m T$ is constant, where λ_m is the wavelength at which the blackbody radiation intensity is maximal and T is the temperature of the blackbody. More specifically,

$$\lambda_m T = 2.898 \times 10^{-3}\ \mathrm{m \cdot K} = \frac{hc}{4.96511\ k_B}.$$

where 4.96511 is the root of the equation $e^x(5-x) = 5$. If $\lambda_m = 490$ nm then $T_{\odot}^{sur} \approx 5900^{\circ}$K.

So we go from a temperature of $T_{\odot}^{int} = 1.6 \times 10^{7}$ K at the center to $T_{\odot}^{sur} = 5900$ K at the surface. If the gradient is constant, it is

$$\frac{dT}{dx} = \frac{T_{\odot}^{int} - T_{\odot}^{sur}}{R_{\odot}} = 0.023\,\mathrm{K/m}$$

This temperature gradient is maintained and further gravitational collapse is prevented by the radiation pressure exerted by the energy issuing from the fusion processes in the interior.

---

2. List three radioactive nuclides that naturally occur (in easily detectable amounts) on Earth today. What are the half-lives of these nuclides? How did they originate? At least one of them should have a half-life of less than a thousand years. Explain how it is possible to find nuclides with a half-life so much shorter than the age of the Earth.

Several nuclides have lifetimes of the order of 10^9 or 10^{10} years, which is about the age of the earth, and they are still around today. Among them are Uranium-238 (^{238}U) and Uranium-235 (^{235}U). Many of their decay products are also radioactive and have much shorter lifetimes. For instance:

- Radon-222 (^{222}Rn) is present in the atmosphere and collects in caves and basements. It has a half-life of 3.8 days and is created when (^{238}U) decays. ^{238}U has a long half-life of 4.51×10^9 years and is found in some rocks. With such a long half-life it serves a slow but steady source of ^{222}Rn.
- Iodine-131 (^{131}I) which has a half-life of 8.02 days is produced when ^{235}U fissions. While this isotope occurs naturally, most of it is man-made.

Carbon-14 (^{14}C) is also present in the earth's atmosphere; it has a half-life of 5730 years. It is created when cosmic rays enter the atmosphere producing a shower of particles, some of which are neutrons. Nitrogen (^{14}N) can capture a neutron and give off a proton, leading to ^{14}C.

---

3. Particle decays.

(a) Name the particles involved in the process $n \to p + e + \bar{\nu}_e$. Which are hadron, mesons, baryons, leptons? Complete the following table

	n	p	e	$\bar{\nu}_e$
Electric charge				
Baryon number				
Electron number				

(b) Some of the particles in this process are made out of quarks. Describe the process in terms of the transformation of the constituent quarks.

(c) What are these decays forbidden?

i. $n \to \pi^+ + e^- + \bar{\nu}_e$

ii. $n \to e^+ + e^-$

iii. $n \to p + \mu^- + \bar{\nu}_\mu$

iv. $n \to \bar{p} + e^+ + \nu_e$.

(a) The particles in the process $n \to p + e + \bar{\nu}_e$ are the neutron (n), the proton (p), the electron (e), and the electron antineutrino ($\bar{\nu}_e$). Hadrons are made of quarks that are held together by the strong interaction (that is, by exchanging gluons). The neutron and the proton are hadrons. Hadrons come in two varieties: mesons which are made of a quark and an antiquark bound together and baryons which are made of three quarks bound together. (So what? A meson has integer spin and is bosonic, while a baryon has half-integer spin and is fermionic.) The neutron and proton are baryons. The electron and electron antineutrino are leptons.

	n	p	e	$\bar{\nu}_e$
Electric charge	0	$+$	$-$	0
Baryon number	+1	+1	0	0
Electron number	0	0	+1	-1

(b) The neutron and proton are baryons made of up and down quarks, u's and d's. The u has a charge of $+2/3$ and the d has a charge of $-1/3$. From this we can deduce that the neutron, which is neutral, is made of one up and two down quarks, while the proton, which has charge $+1$, is made of one down and two up quarks. Thus the process $n \rightarrow p + e + \bar{\nu}_e$ can be seen as $d \rightarrow u + e + \bar{\nu}_e$ where the d on the left-hand-side is bound to a u and a d, making an n, and the u on the right-hand-side is also bound to a u and a d, this time making a p.

(c)

i. $n \rightarrow \pi^+ + e^- + \bar{\nu}_e$ violates baryon number conservation.

ii. $n \rightarrow e^+ + e^-$ also violates baryon number conservation.

iii. $n \rightarrow p + \mu^- + \bar{\nu}_\mu$. Charge is conserved; baryon number is conserved; muon number is conserved. However, in the rest frame of the neutron, it has an energy 939.573 MeV, while the proton has energy 938.280 MeV and the muon 105.7 MeV. Thus unless something were to bash into the neutron and give it sufficient energy the above process could not take place (except perhaps virtually) because of energy Conservation.

iv. $n \rightarrow \bar{p} + e^+ + \nu_e$ violates baryon number conservation.

---

4. Smoke detectors. Read about them in *Bloomfield*. The one we looked at in class uses an ^{241}Am source with an activity of 1 μ Ci $= 37$ **kBq**.

(a) Find out something more about ^{241}Am. How is it manufactured? Is it cheap and plentiful? How does it decay, mostly?

(b) How many decays per second is 1 μCi? How much ^{241}Am, in grams, gives out 1 μCi? After one year, how much is the activity of the decay product(s), compared to that of the remaining americium? (Is it significant?) Why is it important that the isotope used in a smoke detector not decay into radioactive gas such as radon?

(c) If you kept 1 μCi of ^{241}Am in your mouth for an hour, would you get a bad dose of radiation? You can find out in Melissinos what is the permissible quantity of an ingested isotope, but here we have to worry about a local burn, rather than a whole-body effect.

(a) I found my information on ^{241}Am from the webpages at
http://www.uic.com.au/nip35.htm
and
http://www.in-search-of.com/frames/periodic/elements/95.html

One starts with $^{238}_{92}$U in a nuclear reactor. It captures a neutron giving $^{239}_{92}$U, which in turn beta decays (i.e gives off an electron) yielding $^{239}_{93}$Np, which beta decays again leading to $^{239}_{94}$Pu. This isotope of plutonium captures a neutron ($^{240}_{94}$Pu) and then another producing $^{241}_{94}$Pu. Finally this last isotope of plutonium beta decays to yield $^{241}_{95}$Am. According to the first website above, a gram of Americium oxide, AmO_2 sells for \$1500 (but this amount is enough to make 5000 smoke detectors). ${}^{241}_{95}$Am alpha decays, which leads to ${}^{237}_{93}$Np, which has a half life of 2.14×10^6 years, and thus is relatively stable compared to ${}^{241}_{95}$Am which has a half life of 432 years.

(b) A becquerel (1 Bq) is a decay per second $\mathrm{decay/sec}$. So

$$1\ \mu\mathrm{Ci} = 3.7 \times 10^4\ \mathrm{Bq} = 3.7 \times 10^4\ \mathrm{decays/sec}.$$

The number of decays per unit time is proportional to the number of particles at the particular time in question

$$\frac{dN}{dt} = -\frac{1}{\tau}N,$$

where τ is the constant of proportionality and has dimensions of time. The minus sign appears because the number of particles is decreasing. The solution of this differential equation is

$$N(t) = N_0 \mathrm{e}^{-t/\tau},$$

where N_0 is the initial number of particles. We know the half-life $t_{1/2}$ (the time at which the number of particles, in this case ${}^{241}_{95}$Am, is half of its initial value is 432 years, so

$$N(t_{1/2}) = \frac{N_0}{2} = N_0 \mathrm{e}^{-t_{1/2}/\tau},$$

or

$$\tau = \frac{t_{1/2}}{\ln 2} = \frac{t_{1/2}}{0.693}.$$

So $\tau = 623$ years. The rate of particles undergoing decay is

$$\frac{dN}{dt} = -\frac{N_0}{\tau}\mathrm{e}^{-t/\tau}.$$

We want the decay rate (at least initially) to be 3.7×10^4 decays/sec, so

$$\begin{aligned} 3.7 \times 10^4\ \mathrm{decays/sec} &= \frac{N_0}{623\ \text{years}}\left(\frac{1\ \text{year}}{365.24\ \text{days}}\right)\left(\frac{1\ \text{day}}{24\ \text{hours}}\right)\left(\frac{1\ \text{hour}}{60\ \text{minutes}}\right)\left(\frac{1\ \text{minute}}{60\ \text{seconds}}\right) \\ &= \frac{N_0}{1.965 \times 10^{10}\ \text{sec}}. \end{aligned}$$

Therefore $N_0 = 7.27 \times 10^{14}$ particles which corresponds to

$$7.27 \times 10^{14}\,\text{particles}\left(\frac{1\ \text{mole}}{6.022\ \times 10^{23}\ \text{particles}}\right)\left(\frac{273\ \text{gm AmO}_2}{1\ \text{mole}}\right) = 3.3 \times 10^{-7}\ \text{gm}.$$

Recall that $1\ \mu$Ci is the activity of $1\ \mu$gm of radium (and its daughters). Because Am has a shorter half life (the half life of ^{226}Ra is 1.6×10^4 years), it requires less of it to have the same activity.

The activity (dN/dt) after one year is

$$\frac{dN(1\ \text{year})}{dt} = -\frac{N_0}{\tau}\exp(-1\ \text{year}/623\ \text{year}) = -0.9984\frac{N_0}{\tau},$$

that is, the activity is 99.8% of its initial value, hardly changed at all.

In terms of the operation of the smoke detector, a reaction in which the products are themselves radioactive is not a problem. Of the decay products, it is the alpha particles that ionize the air which carry the current in the detector. So we want a reaction or series of reactions in which alphas are maximized and gammas are minimized. If the further decay of the products increased the relative alpha yield, it would be useful, and we could simply reduce the amount of material we use. But if the further decay increases the amount of gamma radiation, it is undesirable. Furthermore, it is desirable that any decay products remain chemical inert and in the solid form. Radon is particularly bad because it is a gas.

(c) According to the first webpage above "Americium-241 is ... a potentially dangerous isotope if it is taken into the body in soluble form. It decays by both alpha activity and gamma emissions and it would concentrate in the skeleton. However, swallowing the radioactive material from a smoke detector would not lead to significant internal absorption of Am-241, since the dioxide is insoluble. It will pass through the digestive tract, without delivering a significant dose."

Melissinos (p. 190) gives a conversion of Ci to $\mathrm{rem/hr}$ for ^{60}Co valid if the tissue in question is $1\ \mathrm{m}$ from the source

$$1\ \mathrm{Ci\ of\ }^{60}Co\ \mathrm{at}\ 1\ \mathrm{m} = 1.3\ \mathrm{rem/hr}.$$

Since the source is in the mouth and we are interested to damage to the mouth, we will use the distance of $1\ \mathrm{cm}$. The intensity varies like $1/r^2$, so the conversion will have a factor of 10^4 because of the closer proximity. So the conversion for a microgram of ^{60}Co is

$$1\ \mu\mathrm{Ci\ of\ }^{60}Co\ \mathrm{at}\ 1\ \mathrm{cm} = 0.013\ \mathrm{rem/hr}.$$

Now let us assume a similar conversion for ^{241}Am; this is probably overkill as the gamma rays in the decay of Am are low energy. Nevertheless, exposure for an hour would yield $0.013\ \mathrm{rem}$. This is fairly low; however, there may still be some local burning.

---

5. Expansion of the universe.

(a) Does the expansion of the universe imply that the earth is getting bigger? The solar system? Our galaxy? The distance from our galaxy to the Virgo cluster?

(b) If Hubble's constant is $70\ \mathrm{km/sec/Mpc}$, how far are galaxies that are moving away from us at one third the speed of light? At even greater distances, the galaxies eventually move away from us at a speed exceeding the speed of light, if Hubble's law is valid. Is this possible?

(c) If Hubble's constant is $70\ \mathrm{km/sec/Mpc}$, how old is the universe in the simplest model in which the "radius" increases in proportion to $t^{2/3}$? Justify your answer.

(a) The size of the earth is determined by the kind and amount of materials it is made of and the various processes it is undergoing. Its size is locally determined and is unaffected by the expansion of the universe. The same is true of the solar system and galaxy, where the size is determined by the gravitational attraction of the constituent bodies. The distance from our galaxy to the Virgo cluster is a another story. They are not bound together by gravity, and thus the distance between them grows as the universe expands. In fact, this distance was among the original measurements of Hubble which gave credence to the idea of an expanding universe.

(b) Hubble found that on average the galaxies were moving away from ours with velocities proportional to the distances to them

$$v = H_0 R,$$

where $H_0 = 70\ \mathrm{km/sec/Mpc}$. According to this relation, the distance to a galaxy moving away with one third the speed of light is

$$R = \frac{v}{H_0} = \frac{1 \times 10^8\ \mathrm{m/sec}}{7 \times 10^4\ \mathrm{m/sec/Mpc}} = 1.4 \times 10^3\ \mathrm{Mpc}.$$

(This is close to the size of the known universe.)

An object cannot move through space at a speed faster than that of light, but two objects can move away from each other at speeds exceeding the speed of light if the space between them is expanding. The analogy is ants crawling on a balloon while it is being blown up. There is a maximum speed at which the ants can crawl with respect to the balloon, but the distance separating the ants can grow faster. Note that while it is possible for an object in an expanding universe to move away with a speed greater than that of light, it is impossible for us to *see* it.

(c) The first thing to note is that Hubble's constant is not constant. It varies with time. If we note that $v = dR/dt$, we have

$$\frac{dR}{dt} = H(t)R,$$

or

$$\frac{dR}{R} = H(t)dt,$$

which has a solution

$$R(t) = R(t_0)\exp\left[\int_{t_0}^{t} H(s)\ ds\right].$$

By using a simple "escape velocity" criterion, one is lead to a another relationship between R and t:

$$\frac{R(t)}{R(t_0)} = \left(\frac{t}{t_0}\right)^{2/3}.$$

See the handout on "The Thermal History of the Universe". These two formulas can be reconciled if

$$H(t) = \frac{2}{3\,t}.$$

Note that $H(t)$ has dimensions of reciprocal time.

If we know the Hubble "constant" for the present, we can calculate from it t

$$t = \frac{2}{3\,H(t)}.$$

Now all we have to do is transform H from these strange units in which it is given. One parsec is

$$1\ \text{parsec} = 3.26\ \text{light year},$$

so

$$H = \frac{7 \times 10^4 \text{ m/sec}}{10^6 \cdot 3.26 \cdot 3 \times 10^8 \text{ m/sec} \cdot \text{yr}} = 7.16 \times 10^{-11} \text{ yr}.$$

Thus

$$t = \frac{2}{3\,H} = 9.31 \times 10^9 \text{ yr}.$$