Physics 312 – Assignment 2

1. A parallel plate capacitor has a small hemispherical boss of radius a on one of its plates. To be definite, you can assume that A is the area of each plate, d the separation between them, with $\sqrt{A} \gg d \gg a$; take the plates to be perpendicular to the z axis and put the boss at the center of the lower plate. However, the results you will obtain hold more generally, as long as a is much smaller than the distance from the boss to the closest edge. The problem is to find the electric field.

You can imagine that the lower plate is a flat plain, the upper plate is the bottom of a large cloud, and the hemispherical boss represents a hill in the middle of the plain, or maybe a tree in the middle of a field. The problem then tells you something about the chances of being struck by lightning in hilly country, or why isolated trees are at risk of being struck.

(a) If σ_0 is the charge density on the lower plate far away from the boss, what is the electric field far away from the boss, E_0? You can (approximately) relate σ_0 to the area, A, and to the total charge on the plate, Q, but it is not necessary to do so.

From Gauss' Law we find $\sigma_0 = E_0\varepsilon_0$, (in SI units, where $\varepsilon_0 = 8.85\times10^{-12}\mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$) or $\sigma_0 = 4\pi E_0$ (in Gauss – cgs units). See for example *Serway*, page 481, or *Tipler*, page 647. Hence $E_0 = \sigma_0/\varepsilon_0$ (SI) or $E_0 = 4\pi\sigma_0$ (Gauss – cgs), is the field on the lower plate, far from the boss. It is also true that $\sigma_0 = Q/A +$ corrections of order a^2/A.

(b) Find the field and the charge density everywhere on the lower plate.

This problem is equivalent to that of a conducting sphere in a uniform applied field. It is also similar, mathematically, to the problem of fluid flow past a sphere, for a non-viscous, incompressible fluid. (See last semester's notes for the very similar problem of flow past a cylinder: http://www.phys.virginia.edu/classes/311/notes/fluids11/node19.html). Anyhow, the hint was given in class to find first the electrostatic potential Φ and then use $\mathbf{E} = -\nabla\Phi$. In empty space Φ satisfies the Laplace equation $\nabla^2\Phi = 0$, and on the surface of the conducting plate Φ must be constant. Since Φ is defined up to a constant, one can set $\Phi = 0$ on the lower plate. The hint was given to look for a solution of the form

$$\Phi = A_1\, z + B_1\, z/r^3 = \left(A_1\, r + B_1/r^2\right)\cos\theta.$$

One can verify that this expression satisfies the Laplace equation expressed in spherical coordinates

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0.$$

The problem then is to determine A_1 and B_1 using the data of the problem:

(i) For large r, the field must be $E_0\hat{\mathbf{z}}$. Since Φ reduces to $A_1\, z$ for large r, we see that

$A_1 = -E_0$.

(ii) Φ clearly vanishes on the plane $z = 0$. It must also vanish on the boss, i.e., for $r = a$ and $\theta \le \pi/2$. This gives

$$-E_0\, a + B_1/a^2 = 0$$

Hence $B_1 = E_0\, a^3$ and we find

$$\Phi = -E_0 \left(1 - \frac{a^3}{r^3}\right) z = -E_0 \left(r - \frac{a^3}{r^2}\right) \cos\theta \tag{1}$$

The electric field is obtained from the electrostatic potential by taking a gradient: $\mathbf{E} = -\nabla\Phi$. The field on the flat part of the plate is most easily calculated in Cartesian coordinates. Recall that the gradient of a function points in the direction in which that function changes most rapidly (*Tipler*, p. 671, *Serway*, p. 716) and that Φ is constant in the plate, *i.e.* it does not vary with x or y in the flat part. Thus the gradient of Φ at the surface of the plate ($z = 0$) is in the $\hat{\mathbf{z}}$ direction, with

$$E_z = -\left.\frac{\partial\Phi}{\partial z}\right|_{z=0} = E_0 \left(1 - \frac{a^3}{r^3}\right), \tag{2}$$

which is valid for $r \ge a$. The derivative is computed at constant x and y, and so the first expression in eq. (1) is convenient. (The derivatives with respect to x and y, *i.e.* the components of gradient in the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions, vanish when the expression is evaluated at $z = 0$.)

The field on the boss is most easily calculated in spherical coordinates. Again Φ is constant in the plate, that is, it does not vary with the angles θ or ϕ. Thus the gradient of Φ is in the $\hat{\mathbf{r}}$ direction, with

$$E_r = -\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = 3E_0 \cos\theta, \tag{3}$$

valid for $\theta \le \pi/2$. The derivative is computed at constant θ and ϕ, and so the second expression in eq. (1) is convenient. (The derivative with respect to ϕ is zero, and the derivative with respect to θ vanishes when the expression is evaluated at $r = a$.)

Everywhere on the plate the charge density is $E\varepsilon_0$ (in SI), or $E/4\pi$ (in Gauss – cgs).

(c) Where on the lower plate does the maximum electric field occur, and how much larger than E_0 is it?

Expression (2) varies from E_0 at $r \to \infty$ to 0 at $r = a$, and expression (3) varies from $3E_0$ at $\theta = 0$ to 0 at $\theta = \pi/2$. Hence, the maximum field is $3E_0$, at the top ($\theta = 0$).

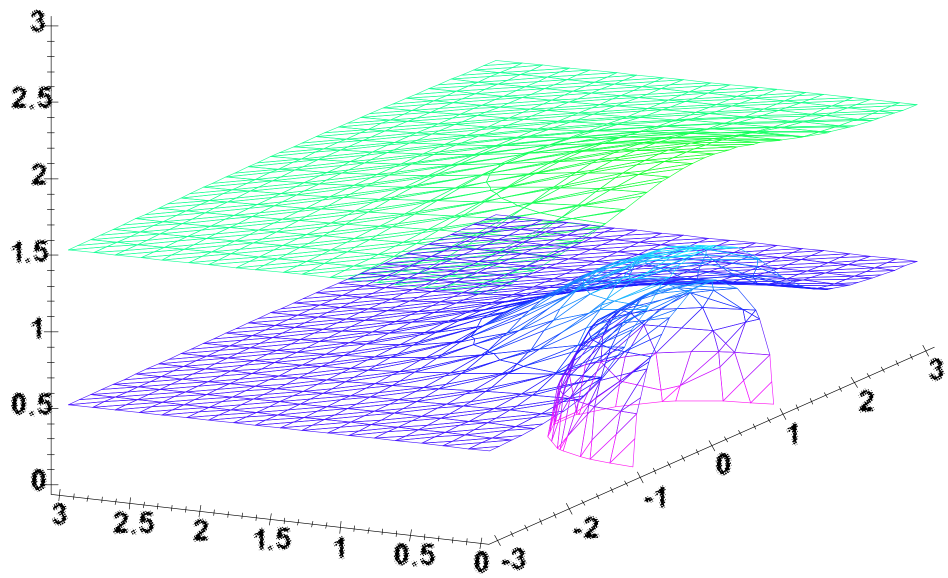
(d) Where on the lower plate does the minimum electric field occur, and how much smaller than E_0 is it?

The minimum field is zero, at the foot of the boss. Hilltops are to be avoided during thunderstorms; it is best to be at the foot of the hill (provided the top is a safe distance away).

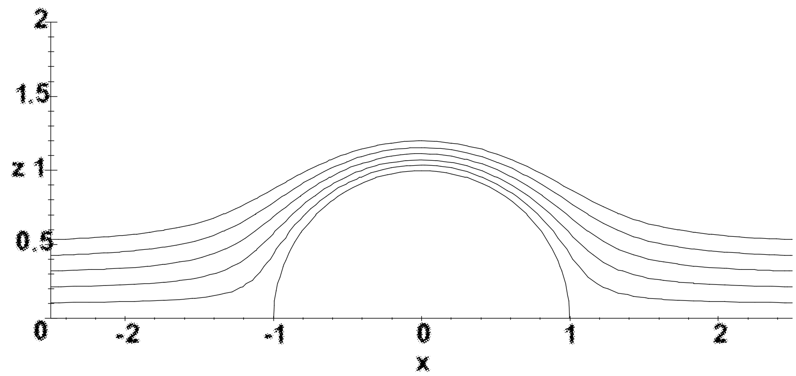
(e) Using a computer program, draw a set of equipotentials, starting from the lower plate. The graph should illustrate how the equipotentials become smooth for $z \gg a$.

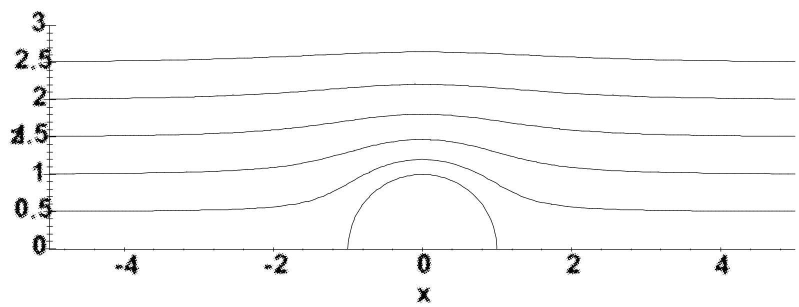
We plot the dimensionless ratio $\Phi/E_0 a$ as a function of distance divided by a, i.e., we use $E_0 a$ as the unit of potential and a as the unit of length.

Here is a view of the boss and of the equipotentials $-\Phi = z\left(1 - \dfrac{1}{(x^2+y^2+z^2)^{3/2}}\right)$ for $-\Phi = 0.5$ and 1.5. where we have taken $a = 1$.



Here are plots of intersections of equipotentials with the xz plane, $-\Phi = z\left(1 - \frac{1}{(x^2+z^2)^{3/2}}\right)$ for $-\Phi = 0.1$ up to 0.5 (top panel) and $-\Phi = 0.5$ up to 2.5.





---

2. Sparks and discharges (as seen in "Jacob's ladder") emit bluish light. What are the frequency and wavelength of this light? What is the energy of an emitted photon in eV? You can give a range of values or a typical value.

Blue light has a wavelength ranging roughly from 4.8×10^{-7} m to 4.2×10^{-7} m. The corresponding frequency range is 6.3×10^{14} Hz to 7.1×10^{14} Hz ($f = c/\lambda$).

The energy of a photon is given by $E = hf = \hbar\omega$ (*Tipler*, p. 1148) where h is Planck's constant ($h = 4.136 \times 10^{-15}$ eV $\cdot$ s). Thus the associated energy range is 2.6 eV to 2.9 eV.

---

3. A steel wire of radius a hangs at the center of a steel can of radius b. If the wire is at potential V and the pipe is grounded, what is the field at the surface of the wire? I am asking for an exact answer in the idealized case of wire and can of infinite length. Assume that the radius of the can is 5 cm, what practically feasible a and V will give a good corona discharge? You can just give typical values.

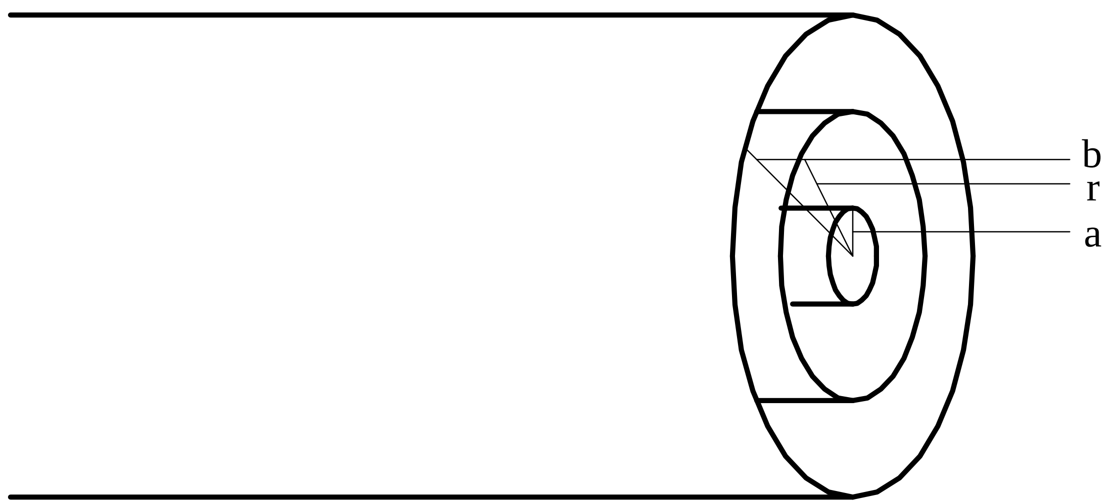


Figure 1:

Applying Gauss' law

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{inside}}}{\varepsilon_0} \tag{4}$$

to the cylindrical geometry, we find

$$E(r)\, 2\pi r L = \frac{Q_{\text{inside}}}{\varepsilon_0}, \tag{5}$$

where r is the radius of the enclosing surface (with $a \leq r \leq b$) and L is its length. (In the infinitely long case, the electric field points radially, so there is no contribution to the integral above from the end caps of the cylindrical surface.) Solving for $E(r)$ gives

$$E(r) = \frac{\lambda}{2\pi\varepsilon_0 r}, \tag{6}$$

where $\lambda = Q_{\rm inside}/L$, the linear charge density of the wire.

But we do not know the linear charge density; we know the applied voltage. We can calculate the potential difference from the electric field through

$$\begin{aligned} V_b - V_a &= -\int_a^b \mathbf{E}(\mathbf{r}) \cdot d\mathbf{r} \\ &= -\frac{\lambda \ln(b/a)}{2\pi\varepsilon_0}. \end{aligned}$$

Hence, the linear charge density is

$$\lambda = -\frac{2\pi\varepsilon_0 \mathcal{V}}{\ln(b/a)}, \tag{7}$$

where $\mathcal{V} = V_b - V_a$. Substituting this expression for λ into the electric field gives

$$E(r) = -\frac{\mathcal{V}}{\ln(b/a)\, r}. \tag{8}$$

The electric field at the surface of the wire is $E(a) = \mathcal{V}/\ln(b/a)a$.

Dielectric breakdown in air occurs at an electric field of roughly $3 \times 10^6\ \mathrm{V/m}$ (*Tipler*, p. 677). It occurs when the neutral molecules become ionized, then there are free charges about to conduct electricity. Some ions are around before breakdown, but at breakdown these ions gain enough energy as they are accelerated by the electric field that when they collide with a neutral atom they cause it to ionize as well—leading to a "chain reaction."

We don't want breakdown to occur over a wide region of our system (because that will cause a spark). We only want breakdown over a small region surrounding the wire, which will serve as a source of ions that spray into the rest space but don't necessarily cause further ionization.

Our first constraint is that the field at $r = a$ must be greater than the breakdown field, i.e.

$$\frac{\mathcal{V}}{\ln(b/a)a} > 3 \times 10^6\ \mathrm{V/m}. \tag{9}$$

The factor $\ln(b/a)$ won't ever be very large. If the ratio $b/a = 100$ then $\ln(b/a) \approx 4.6$; and if $b/a = 1000$ then $\ln(b/a) \approx 6.9$. The important factor is the ratio $\mathcal{V}/a$. If the wire has a radius of $0.5\ \mathrm{mm}$, then $\mathcal{V} > 6900\ \mathrm{V}$. Let us assume $\mathcal{V} = 10,000\ \mathrm{V}$, then at $r \approx 0.72\ \mathrm{mm}$, the field falls below the breakdown threshold. So indeed the dielectric breakdown is only occurring in a small region surrounding the wire.

If there is hot, polluted air rising through the cylinder like the one in this problem, it can be used as a "precipitator." See *Tipler*, p. 682-684 for a short description of an electrostatic precipitator. That section also includes a description of xerography. See also Bloomfield p. 378-383.