Physics 312 – Assignment 5

This assignment contains many short questions and a long one. Several questions are taken from *Bloomfield* Section 11.2, with some changes.

1. (2 points) It is possible to combine the fundamental constants $\hbar$ and e (elementary charge) to obtain a quantity with the dimensions of conductance. Using dimensional analysis, find this "fundamental conductance" and compare it with that of one meter of copper wire having a 1 mm radius, at room temperature. Recall that conductance is defined as the inverse of resistance, $1/R$. Do not confuse conductance with conductivity, or resistance with resistivity.

The problem asks for a combination of $\hbar$ and e with dimensions of conductance; however, the "fundamental conductance" is usually given using h instead of $\hbar$. So your answers may differ by a factor of 2π from what follows.

From $V = IR$ we can determine the dimensions of R knowing that the voltage V has dimension $\mathrm{Joule/Coulomb}$ and the current I has dimension $\mathrm{Coulomb/second}$. Thus we can conclude that R has dimension $\mathrm{Joule \cdot second/Coulomb^2}$. Recalling that h has dimension $\mathrm{Joule \cdot second}$ and e has dimension $\mathrm{Coulomb}$, we see that h/e^2 has the same dimension as resistance R. Finally, the conductance, being the reciprocal of the resistance, has the same dimension as e^2/h. (*This quantity plays a key role in the Quantum Hall Effect, the discovery of which won von Klitzing the Nobel prize in 1985, see Tipler, p. 803-4.*)

The combination h/e^2 has a value

$$\frac{h}{e^2} = \frac{6.626 \times 10^{-34}\ \mathrm{J \cdot s}}{1.602 \times 10^{-19}\ \mathrm{C}} = 2.58 \times 10^4\ \Omega, \tag{1}$$

where Ω stands for ohms. The corresponding conductance is $3.88 \times 10^{-5}/\Omega$. The resistance R is related to the resistivity ρ through

$$R = \frac{\rho L}{A}, \tag{2}$$

where L in the length of the wire and A its cross sectional area. (See *Tipler*, p. 721.) The resistivity of copper is $1.7 \times 10^{-8}\ \Omega \cdot \mathrm{m}$ at $20^\circ\ \mathrm{C}$. (See *Tipler*, p. 722.) Thus, the resistance of the wire is

$$R_{wire} = \frac{(1.7 \times 10^{-8} \Omega \cdot \mathrm{m})(1\ \mathrm{m})}{\pi (1 \times 10^{-3}\ \mathrm{m})^2} = 0.0054\ \Omega, \tag{3}$$

and the corresponding conductance is $185/\Omega$. The resistance of the copper wire is much smaller (about a million times smaller) than the ratio h/e^2; and the conductance of the copper wire is correspondingly about a million times larger than e^2/h.

2. (1 point) A current of 1 Ampere flows in a copper wire. In how many seconds will a billion electrons pass through a given point on the wire?

One ampere corresponds to

$$1\ \mathrm{A} = \left(\frac{1\ \mathrm{C}}{\mathrm{s}}\right)\left(\frac{1\ \mathrm{e}}{1.609 \times 10^{-19}\ \mathrm{C}}\right) = \frac{6.215 \times 10^{18}\ \mathrm{e}}{\mathrm{s}}. \tag{4}$$

Since electrons are carrying the current above, 6.215×10^{18} electrons will flow through a given cross section each second. Thus the time for a billion electrons is

$$\left(1 \times 10^{9}\mathrm{e})\right)\left(\frac{\mathrm{s}}{6.215 \times 10^{18}\ \mathrm{e}}\right) = 1.609 \times 10^{-10}\mathrm{s}, \tag{5}$$

or 0.16 nanoseconds.

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3. (4 points) Thermal energy can shift some of the electrons in a hot semiconductor from valence levels to conduction levels. What effect do these shifts have on the resistivity of the semiconductor? In an intrinsic semiconductor or insulator, the resistivity ρ is found to change by a factor f when the temperature T is reduced from 300° K to 150° K. Is f greater or smaller than 1? What happens to ρ when T is reduced to 75 K? If the band gap is 1.4 eV, compute ρ_{150}/ρ_{300} and ρ_{75}/ρ_{300}.

A useful expression for the resistivity ρ is

$$\rho = \frac{m_e}{ne^2\tau}, \tag{6}$$

where we assume that the current is conducted by electrons and where e is the of charge an electron, m_e the mass of an electron (sometimes it is replaced by what is called "effective mass"), n the *carrier density* i.e. the number of electrons that are "free" to move per unit volume and τ is the relaxation time often thought of as the average time between collisions (which are the source of friction or viscosity). (See *Tipler*, p. 735 or *Serway*, p. 627.)

The main effect of changing the temperature in a conductor is to change τ, but the main effect in a semiconductor is to change n. If thermal energy shifts electrons to the conduction level, it is adding to n and thus lowering the resistivity which varies in inverse proportion.

The resistivity ρ is found to change by a factor f when the temperature T is reduced from 300° K to 150° K, i.e. $\rho_{150} = f\rho_{300}$. Our argument proceeds: the higher the temperature, the more electrons in the conduction band, and the lower the resistivity. Thus, $\rho_{300} < \rho_{150}$ and $f > 1$. If the temperature is reduced further, the resistivity goes up even more.

The carrier density varies roughly as

$$n \propto \exp\left(-\frac{E_g}{2kT}\right), \tag{7}$$

where E_g is the energy gap. (See *Melissinos*, p. 8 or the lecture notes on "Carrier Density.")

So the resistivities are affected as follows

$$\frac{\rho_{150}}{\rho_{300}} \approx \frac{n_{300}}{n_{150}} \approx \exp\left(-E_g/2k\,(300^\circ\mathrm{K}) + E_g/2k\,(150^\circ\mathrm{K})\right), \tag{8}$$

so that

$$\frac{\rho_{150}}{\rho_{300}} \approx \mathrm{e}^{27} \approx 5.3 \times 10^{11}. \tag{9}$$

Similarly

$$\frac{\rho_{75}}{\rho_{300}} \approx \mathrm{e}^{81} \approx 1.5 \times 10^{35}. \tag{10}$$

(This last factor seems too high, and one assumes that other effects must enter to lower it.)

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4. (2 points) Based on the result of problem 3, would heating the photoconductor in a xerographic copier improve or diminish its ability to produce sharp, high-contrast images? Explain.

Xerography exploits the fact that some materials are insulating in the dark and conducting when exposed to light: the photons associated with visible light promote electrons from the valence band to the conduction band. If the material becomes sufficiently hot, a significant number of electrons may be excited into the conduction band by thermal energy. Thus the material would be a slightly better conductor in the dark, and charges in the dark regions (which we want to remain fixed to the drum to attract toner in the next stage of the process) might leak away. Thus heating the photoconductor would diminish the ability to create a sharp image.

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5. (2 points) What is the maximum energy of a photon of visible light? How does it compare with the typical band gap of (a) insulators and (b) semiconductors? How does this explain the fact that typical semiconductors are opaque, while insulators can be clear or translucent?

The wavelength λ of visible light ranges from roughly $400\,\mathrm{nm}$ to $700\,\mathrm{nm}$. The energy of a photon is given by

$$E = hf = \frac{hc}{\lambda}, \tag{11}$$

and so the energy of photons of visible light ranges from 4.97×10^{-19} J (3.10 eV) to 2.84×10^{-19} J (1.77 eV).

The band gap of carbon (C) in the diamond structure, which is an insulator, is 7 eV (*Tipler, Modern Physics*, p. 336). The gap energy is larger than the energy of a photon in the visible range, and photons therefore do not excite electrons from the valence band to the conduction band. With no energy levels roughly hf away, most of the photons pass through unabsorbed and hence diamond is clear.

The semiconductor silicon (Si) has an energy gap of about 1 eV and germanium (Ge) has a gap of around 0.7 eV (*Tipler, Modern Physics*, p.337). The gap energies are less than the energies of photons in the visible range. Therefore, light can excite electrons in the

valence band into the conduction band, and so it does not pass through the material but is absorbed (and later emitted). Light doesn't pass through and the object is opaque.

Materials that are even better conductors are shiny. Why?

Just for comparison kT *at* $T = 300\ K$ *is* 0.026 eV $\approx \frac{1}{40}$ eV.

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6. (2 points) Actually, good insulators can be opaque. Some, like dry wood or rubber, consist of large molecules. Can you explain, qualitatively, how a solid consisting of large molecules, or more generally with a complex structure, can be both opaque and insulating?

In the answer to the previous question we focused on the band gap energies, but there are other energies or rather energy differences around, especially in substances which don't form crystals. If there are energies at or below the energy of visible photons, then photons will be absorbed. (There are a few provisos, you can't always get from one state to another via a photon.) Organic molecules often have energies associated with vibrational modes in the infrared region (which is why infrared spectroscopy is a useful tool in organic chemistry). Thus light is absorbed, and the materials are opaque. Since these excitations are not associated with the conduction band, the material may be opaque and still an insulator.

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7. (extra points given for good long answers) (a) (1 point) Why do incandescent lamps often burn out just after they are switched on? (b) (1 point) Why do old bulbs show a dark spot at the top (rather than somewhere on the side)? See *Bloomfield*, section 6-3.

(a) Suppose a thin spot developed in the filament; that length would have a higher resistance. It would not change the overall resistance of the long filament by much, so the current would not change much. But the same current flowing through a greater resistance will lead to greater dissipation, i.e. more heat. The segment would have a higher temperature, and hence more sublimation, and so grow even thinner. So a thin spot grows thinner and eventually breaks.

However, the above scenario does not explain why bulbs tend to burn out more when first turned on. The bulb is going through some dramatic changes when first turned on. For example, its temperature is changing by thousands of degrees. (According to *Bloomfield* (p. 276), they operate at about 2500°K.) One consequence is that the resistivity changes with temperature—it is $5.44 \times 10^{-8}\ \Omega$m at 300°K, $21.5 \times 10^{-8}\ \Omega$m at 900°K, etc. Hence the bulb draws a lot more current when initially turned on. As discussed above the local temperature of the thin spot is higher, and perhaps with the large initial current it gets hot enough to melt.

In a different scenario, one concentrates on the fact that the wire is not one big perfect crystal (i.e. the atoms are not arranged in a perfect lattice), instead there are grains in which the lattices have different orientations with "domain walls" separating them. Each time the wire is heated up, these grains grow larger. Eventually the grains reach the size of the diameter of the wire, then the domain wall spans the wire. Another consequence

of the temperature change is that it will undergo some thermal expansion (the coefficient for linear expansion of tungsten (W) at 25°K is 4.5×10^6 K^{-1}, CRC Handbook), and in this picture the domain wall is a crack waiting to happen when the wire is under "thermal stress."

There are other scenarios.

(b) According to *Bloomfield* (p. 279), bits of tungsten which have come off of the hot filament are carried by convection currents. The gas nearest the filament is the hottest and therefore the lightest; it is carried upward. There it cools and begins to descend. Some of the tungsten condenses at the top of the bulb. According to this explanation, the dark spot occurs at the "top" of the bulb, which may be at different parts of bulb depending on how it is situated. (In some fixtures the bulbs lie sideways or even upsidedown, and the spot there should occur in a different place on the bulb.)

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8. (5 points and possibly extra points) (a) Plot the potential $V = x^2/(1 + x^5/100)$ for x between -2 and +10 and find its maximum value inside this interval (endpoints excluded). (b) Integrate numerically the Schrödinger equation

$$-\frac{d^2y}{dx^2} + Vy = Ey$$

for several values of E between 0 and the maximum of V. Start the integration at $x = -2$, assuming that the total potential is infinite for $x < -2$. Find the approximate ground state.

Plot $V(x) = x^2/(1 + x^5/100)$

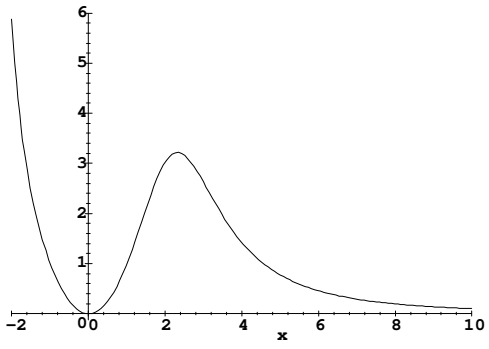

Figure 1:

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By setting the derivative of $V(x)$ to zero to find its maxima and minima:

$$V'(x) = \frac{x(2 - 3x^5/100)}{(1 + x^5/100)^2} = 0, \tag{12}$$

which has solutions $x = 0$ which is a minimum and $x = (200/3)^{1/5} \approx 2.316$ which is a maximum. The height of the well is $V(2.316) \approx 3.219$.

Now, we apply a "shooting" algorithm to find states which have larger amplitudes in the "well" (the region $-2 < x < 2.316$) than they do outside the well ($x > 2.316$). Since the potential $V(x)$ is infinite at $x = -2$ the particle cannot be there at all, and therefore the probability amplitude $y(x)$ must be equal to zero there. So we insist $y(-2) = 0$. We give the wavefunction an initial slope $y'(-2) = c$—I chose $y'(-2) = 0.1$ (Because the differential equation is linear, if $y(x)$ is a solution, so is $ay(x)$. Therefore when we select an initial slope, we are just choosing one of the solutions from this infinite family of solutions.) Next we choose an energy E and numerically integrate the equation using MAPLE. As we vary E the resulting wavefunctions $y(x)$ look very different — most have a large amplitudes outside the well, but close to select energies the amplitudes in the well are larger. The latter are the wavefunctions we are searching for.

I started with $E = 0.1$, looked at $y(x)$, increased the energy by 0.1, looked at $y(x)$, and so on. As the energy increased, the amplitude outside the well decreased. A change occurred between $E = 1.0$ and $E = 1.1$: the first peak outside the well was positive for $E = 1.0$ and was negative for $E = 1.1$. So I stepped through this region more carefully, incrementing the energy by 0.01. The wavefunction with the largest amplitude inside the well was for $E = 1.02$, to two decimal places.

A trick for estimating energies in wells is to approximate the potential well $V(x)$ near its minimum x_0 by a Taylor expansion to order $(x - x_0)^2$. In this particular case, $x_0 = 0$ and one would have $V(x) \approx x^2$. With this approximation we have the Schrödinger equation for the harmonic oscillator that we looked at previously in a problem session. Recall it had an eigenvalue of $E = 1.0$ which is close to the value we found above.

Classically, for energies $0 < E < 3.219$ a particle is either inside the well or outside the well. In quantum mechanics we see that the wavefunction has some weight inside the well, outside the well, and in the classically forbidden region. Outside the well the wavefunction oscillates almost periodically. Let us compare the value of $|y(x)|^2$ at its maximum inside the well to one of its maxima outside the well, preferably at as large an x as possible. With the choices made above the peak inside the well $y^{\max}_{\text{inside}} \approx 0.233$ while the peak out near $x = 10$ was $y^{\max}_{\text{outside}} \approx 0.021$. The tunneling probability is proportional to $|y^{\max}_{\text{outside}}|^2/|y^{\max}_{\text{inside}}|^2$.

George Gamow used a somewhat similar problem to explain the range in lifetimes associated with α decay of radioactive nuclei. See *Tipler, Modern Physics*, p. 231 or http://www.phys.virginia.edu/classes/252/Barriers/Barriers.html.

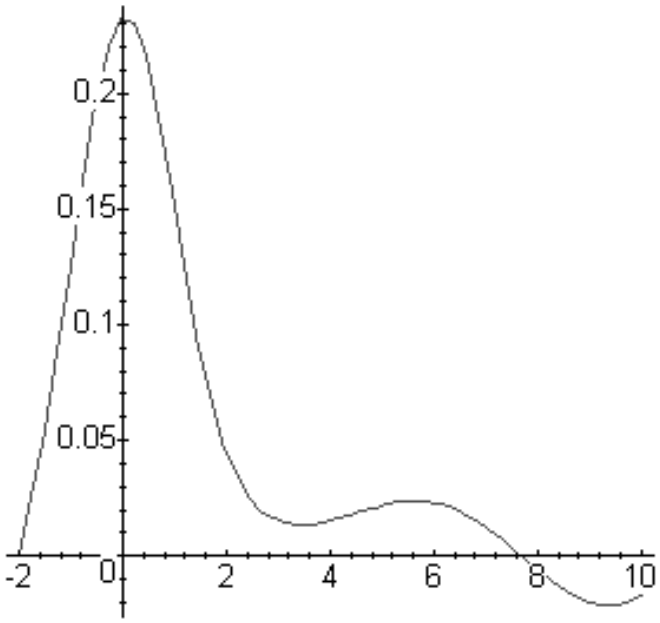

Figure 2: