Physics 312 – Assignment 8

1. Fundamental constants from LED data. In class, we measured the threshold voltage to get appreciable light from various LED's (red, yellow, green, blue). Use these data to obtain a rough estimate of h/e, assuming that all the energy eV gained by an electron passing through the LED is transferred to a single photon. Plot (by hand) the data to check this simple assumption.

The data obtained in class follows

Wavelength (in $10^{-9}m$)	Frequency (in $10^{14}/\sec$)	Voltage (in $volts$)
660	4.54	1.35
630	4.76	1.46
590	5.08	1.58
565	5.31	1.66
450	6.66	2.28

There is a threshold voltage V_{th}, a voltage we have to apply to put electrons into the conduction band. Once electrons are in the conduction band, they can drop into the valence band. When they do this they give off a photon, the energy of which is E_g, the gap energy. So we have

$$hf = E_g = eV_{th} + \mathcal{E},$$

where $\mathcal{E}$ is the difference between the gap energy and the threshold energy. It happens that $\mathcal{E}$ is approximately the same for the various LED's used. Thus we can plot V (in $volts$) versus f (in $10^{14}/\sec$) , the slope is h/e.

The following MAPLE code provides a least-square fit to the data above.

$> \quad \mathbf{with(stats);}$

$$[anova, describe, fit, importdata, random, statevalf, statplots, transform]$$

$> \quad \mathbf{Xvalues := [4.54, 4.76, 5.08, 5.31, 6.66];}$

$$Xvalues := [4.54, 4.76, 5.08, 5.31, 6.66]$$

$> \quad \mathbf{Yvalues := [1.35, 1.46, 1.58, 1.66, 2.28];}$

$$Yvalues := [1.35, 1.46, 1.58, 1.66, 2.28]$$

$> \quad \mathbf{eqfit := fit[leastsquare[[x, y], y = a + b * x, \{a, b\}]]\ ([Xvalues, Yvalues]);}$

$$eqfit := y = -.6498871585\ +\ 1.317341956\ x$$

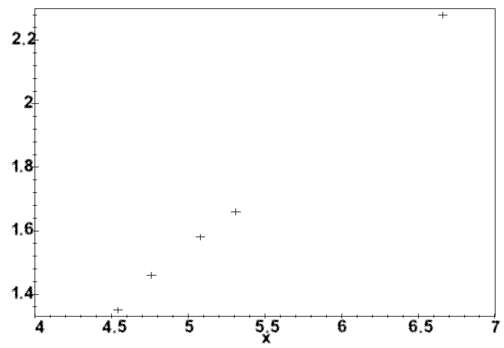


Figure 1:

Reading off the slope from above and recalling that the x axis units are $10^{14}/\mathrm{sec}$, we find $h/e \approx 0.436 \times 10^{-14}$ V$\cdot$sec, which is about 5 percent off the actual value of $h/e \approx 0.414 \times 10^{-14}$ V$\cdot$sec.

The fit can also be done with a spreadsheet (the next version of MAPLE comes with a built-in spreadsheet).

---

2. Circuit parameters. A general circuit is characterized by the three quantities R, C, L.

(a) What are the dimensions of these quantities in the SI? Optional: what are their dimensions in the gaussian system (they are a lot simpler and more intuitive, but in this course we have agreed to use SI).

(b) What are the dimensions of the product LC? What is its physical meaning?

(c) What are the dimensions of the product RC? What is its physical meaning?

(d) What are the dimensions of R/L? What is its physical meaning?

(e) Form a dimensionless combination of R, C, L. What is its physical meaning?

(a) From $V = Ri$, $V = q/C$ and $V = Ldi/dt$, where q is charge and i is current, we can extract the following dimensions

$$dim[R] : \frac{volt}{amp} = \frac{J \cdot s}{C^2} = ohm;$$

$$dim[C] : \frac{coul}{volt} = \frac{C^2}{J} = farad;$$

$$dim[R] : \frac{Volt}{Amp/sec} = \frac{J \cdot s^2}{C^2} = henry.$$

(b) The dimension of LC is s^2. See below for its physical meaning.

(c) The dimension of RC is s.

(d) The dimension of R/L is s^{-1}.

(e) A dimensionless combination is R^2C/L.

If we have a capacitor, a resistor and an inductor in series connected to an ac source, we can model it by the following equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = \mathcal{E}_{max}\, e^{-i\omega t},$$

where ω is the angular frequency of the ac voltage and where we have used $I = dq/dt$ and $dI/dt = d^2q/dt^2$. (The ac source is actually the real part of $\mathcal{E}_{max}\, e^{-i\omega t}$, so we should take the real part of $q(t)$ in the end.) We don't need the ac source for this problem but it occurs in the next problem, so we include it. This differential equation is mathematically equivalent to a driven, damped harmonic oscillator, which has the following equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = \mathcal{F}_{max}\cos(\omega t).$$

The solution to the LRC differential equation is

$$q(t) = \frac{\mathcal{E}_{max}\, e^{-i\omega t}}{L\left(\frac{1}{LC} - \omega^2\right) - iR\omega} + A_+\, e^{-a_+ t} + A_-\, e^{-a_- t},$$

where

$$a_\pm = \frac{\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2},$$

and the constants $A_\pm$ are to be determined from initial conditions.

The first term is called the "steady state" (it oscillates with the driving frequency ω), and the latter two terms are called "transients" (which decay exponentially). The constants $a_\pm$ determine how long the part of the solution involving initial conditions lasts. They are also combinations of the time scales we identified above.

(b′) In the $R \to 0$ limit (which essentially means removing the resistor from the circuit), $a_\pm = \pm i/\sqrt{LC}$. In that case, the latter two terms are oscillatory with the angular frequency $1/\sqrt{LC}$ which is the "natural frequency" of the LC circuit.

(c′) In the $L \to 0$ limit (removing the inductor), $a_- \to 1/RC$, so RC is the decay time of an RC circuit.

(d′) In the $C \to \infty$ limit (removing the capacitor), $a_+ \to R/L$, thus L/R is the decay time associated with an RL circuit.

(e′) We can see from the expression above that $a_\pm$ may be real or complex depending on the sign of discriminant $\left(\frac{R^2}{L^2} - \frac{4}{LC}\right)$. If it is positive, we say the oscillator is "over-damped," and the transients simply decay. If it is negative, we say the oscillator is "under-damped," and there are some oscillations superimposed on the decay of the transients. If it is zero, we say the oscillator is "critically damped." When the dimensionless combination R^2C/L equals 4, the LRC circuit is critically damped. (Note the solution above is not quite right for the critically damped case; it was assumed that $a_+ \neq a_-$ in deriving it.)

Another interpretation of the dimensionless combination found above concerns the quality factor Q of an oscillator. If R is small, $a_\pm \approx -R/2L \pm i/\sqrt{LC}$, meaning that there is a weak decay superimposed on an oscillation. In the undriven case, a little bit of energy is lost with each period $T = 2\pi\sqrt{LC}$. For a damped harmonic oscillator, the quality factor is defined as

$$Q = \frac{2\pi E}{\Delta E_{period}},$$

that is 2π times the ratio of the energy at the beginning of a cycle to the energy lost in that cycle. If only a little energy is lost, the Q factor will be large. Consider when the harmonic oscillator is at its maximum amplitude and all of its energy is potential energy. Then the energy is $\frac{1}{2}kA^2(t)$ where $A(t)$ is the amplitude at time t. The energy a period later is $\frac{1}{2}kA^2(t+T)$. Then

$$Q = 2\pi \frac{\frac{1}{2}kA^2(t)}{\frac{1}{2}k\left[A^2(t) - A^2(t+T)\right]}.$$

We know that when R is small

$$A(t+T) \approx A(t)\, e^{-RT/2L}$$

because the real part of $a_\pm$ is $-R/2L$. So

$$A^2(t) - A^2(t+T) \approx \frac{RT}{L} A^2(t),$$

and

$$Q = 2\pi \frac{L}{RT} = \frac{L\omega_0}{R} = \frac{\sqrt{L}}{R\sqrt{C}}.$$

Therefore, the dimensionless quantity we found above is related to the quality factor of the oscillator:

$$\frac{R^2 C}{L} = \frac{1}{Q^2}$$

---

3. Passive low pass filter: As I mentioned in class, one stage of an archaic AM receiver consists of a low pass filter which removes the RF (radio frequency) carrier wave, leaving the AF (audio frequency) signal. The simplest passive filter (i.e., a filter which does not involve active elements such as transistors), consists of a resistor and a capacitor.

(a) Sketch this filter.

(b) Show that

$$\left|\frac{V_{out}}{V_{in}}\right|^2 = \frac{1}{1 + (\omega RC)^2}.$$

Plot this transfer function and show that it has the right properties to function as a low pass filter.

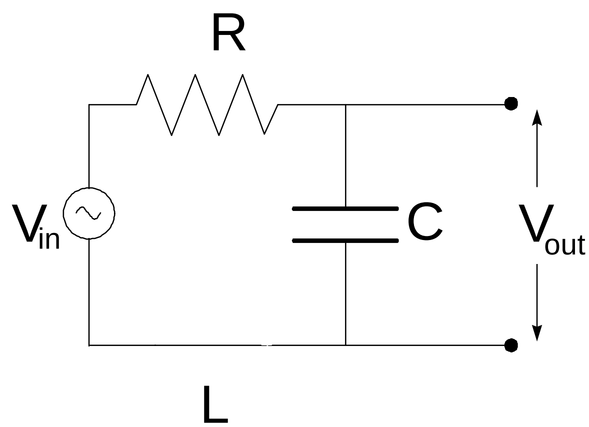
(c) Choose appropriate values of R and C for an AM receiver.

(d) Our simple filter has a very slow "roll-off"; ideally, one would like a "brick wall" transfer function which is 1 up to some frequency ω_0 and zero otherwise. Some improvement is obtained by incorporating an inductor into the filter, as discussed in class. Sketch this filter.

(e) Find the transfer function for this filter. Show that the response at low frequencies is the flattest when $R = \sqrt{2L/C}$. This can be done graphically by plotting the transfer function as a function of a dimensionless frequency for several values of a dimensionless parameter that is proportional to L. Note that for $L = 0$, we are back to the previous case.

(f) How does the transfer function behave at high frequencies?

(a) Below is a sketch of a simple low pass filter.



(b) In Problem 2, we wrote down the solution for the charge on the capacitor in an LRC circuit. To get V_{out}, the voltage across the capacitor, we simply divide by C.

$$V_{out} = \frac{\mathcal{E}_{max} e^{-i\omega t}}{LC\left(\frac{1}{LC} - \omega^2\right) - iRC\omega},$$

where we have dropped the transient part. The transfer function is

$$\left|\frac{V_{out}}{V_{in}}\right|^2$$

if we use the complex notation for the V's. This is the same as

$$\frac{\langle(\mathrm{Re}\, V_{out})^2\rangle_T}{\langle(\mathrm{Re}\, V_{in})^2\rangle_T},$$

where $\langle\rangle_T$ refers to the average over a period of the ac source. If we used the real representation of V, we would have to use the latter expression.

The transfer function for the LRC circuit is then

$$\left|\frac{V_{out}}{V_{in}}\right|^2 = \frac{1}{\left[(1 - LC\omega^2)^2 + R^2C^2\omega^2\right]}.$$

In the case where there is no inductor ($L = 0$), it becomes

$$\left|\frac{V_{out}}{V_{in}}\right|^2 = \frac{1}{[1 + R^2C^2\omega^2]}.$$

A plot of it looks like

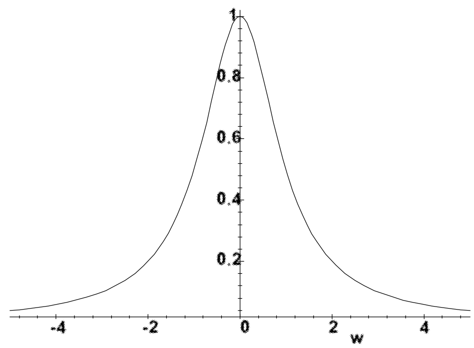


Figure 2:

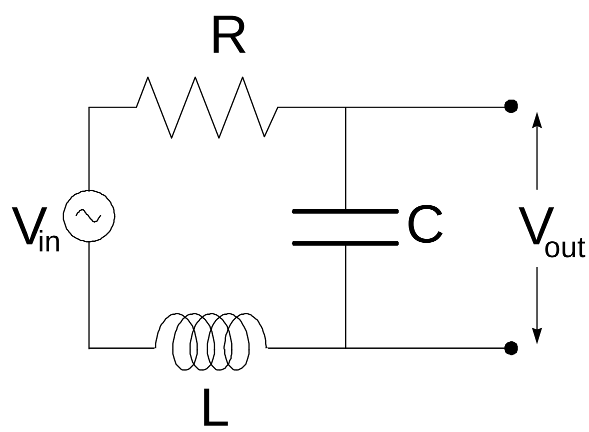
where the x axis is measured in units of $1/RC$, i.e. $w = \omega RC$.

(c) For an AM radio, the desired signal should have frequencies between 30 Hz and $20,000$ Hz which correspond to frequencies we can hear (*Bloomfield*, p. 347), and the carrier frequency is between 550 kHz and 1600 kHz, which are the AM radio frequencies (*Bloomfield*, p. 496). Thus we want our transfer function to be close to one for audible frequencies but to fall to zero for the AM radio frequencies. So if we choose

$$RC = \frac{1}{20,000 \text{ Hz}},$$

then the transfer function is 0.9753 at the highest audible frequency and 0.0496 at the lowest AM radio frequency.

(d) and (e) A plot of the circuit with the inductor follows



And as already calculated above the transfer function is

$$\left|\frac{V_{out}}{V_{in}}\right|^2 = \frac{1}{\left[(1 - LC\omega^2)^2 + R^2C^2\omega^2\right]}.$$

Let us use again the scaled frequency $w = RC\omega$ then

$$\left|\frac{V_{out}}{V_{in}}\right|^2 = \frac{1}{\left[\left(1 - \frac{L}{R^2C}w^2\right)^2 + w^2\right]}.$$

Note the presence of the dimensionless quantity L/R^2C which we identified in Problem 2 with $1/Q^2$, where Q is the quality factor of theLRC circuit. Then

$$\left|\frac{V_{out}}{V_{in}}\right|^2 = \frac{1}{\left[(1 - w^2/Q^2)^2 + w^2\right]}.$$

The desire in a low pass filter is to have the transfer function as flat as possible for the small frequencies and to fall off as rapidly as possible for the undesired high frequencies. Adding the inductor can improve our low pass filter on both scores.

Because the transfer function is even its first derivative is zero at $w = 0$, but we can choose Q to make the second derivative at $w = 0$ zero as well.

$$\frac{\partial^2}{\partial w^2}\left|\frac{V_{out}}{V_{in}}\right|^2\Bigg|_{w=0} = -2 + \frac{4}{Q^2}.$$

So choosing $1/Q^2 = L/R^2C = 1/2$ makes the transfer function flat at small frequencies.

Let's look at it graphically as well. At $Q = 2,\ 1/Q^2 = 0.25$, the LRC transfer function looks like At $1/Q^2 = 0.5$ At $1/Q^2 = 0.75$

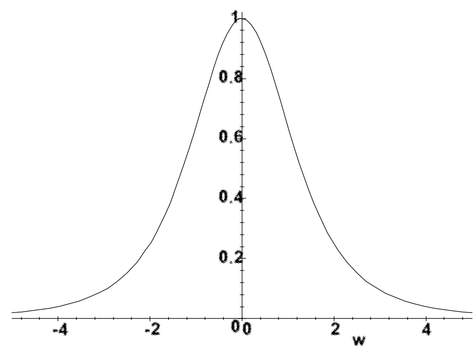


Figure 3:

(f) At large frequencies, the denominator is dominated by the ω^4 term, so that the transfer function varies like ω^{-4} for large ω.

---

4. Using CMOS technology (with enhancement MOSFETs), as in *Bloomfield*, page 481, draw the circuit for an OR gate. Make four replicas of your drawing and indicate explicitly the voltage (or charge) on all the gates and nodes (you may just color the ones that are positive) for the four different possible inputs. Verify that the output is in agreement with the truth table for OR.

The basic building blocks are the p-channel and n-channel MOSFETs in the enhancement mode which are shown below

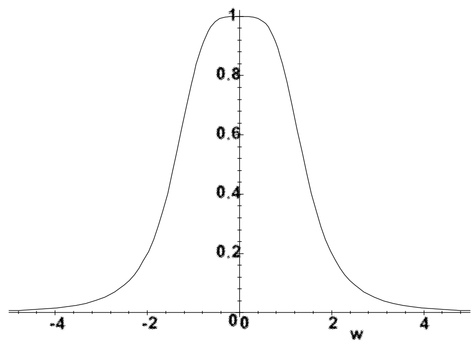
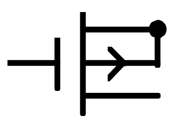


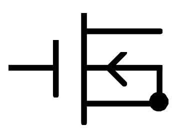
Figure 4:

p-channel MOSFET



- on (i.e. low resistance and small voltage drop) if the gate voltage is negative (low)
- off (i.e. high resistance and large voltage drop) if the gate voltage is positive (high)

n-channel MOSFET



- on (i.e. low resistance and small voltage drop) if the gate voltage is positive (high)
- off (i.e. high resistance and large voltage drop) if the gate voltage is negative (low)

An OR gate can be constructed from four MOSFETs, two n-channel and two p-channel MOSFETs in the following way

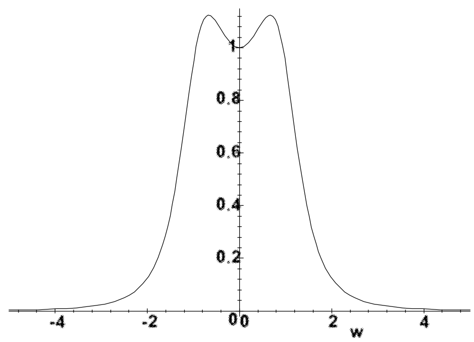
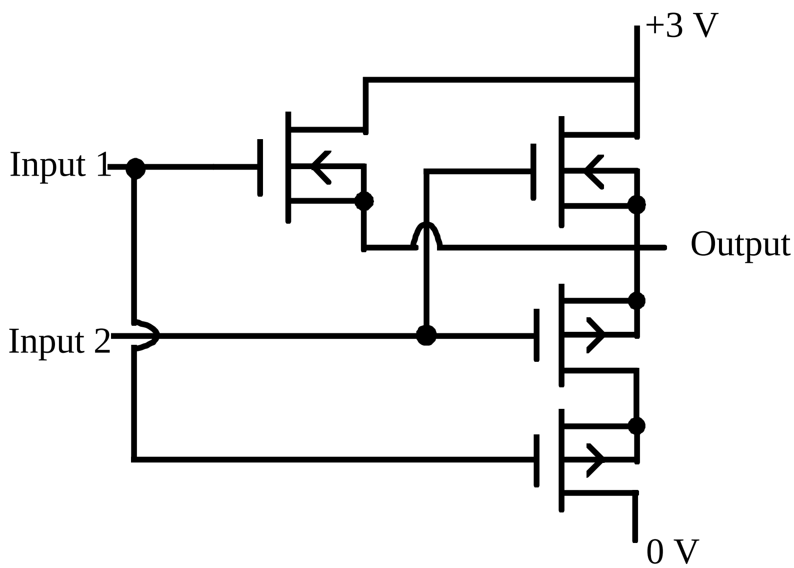
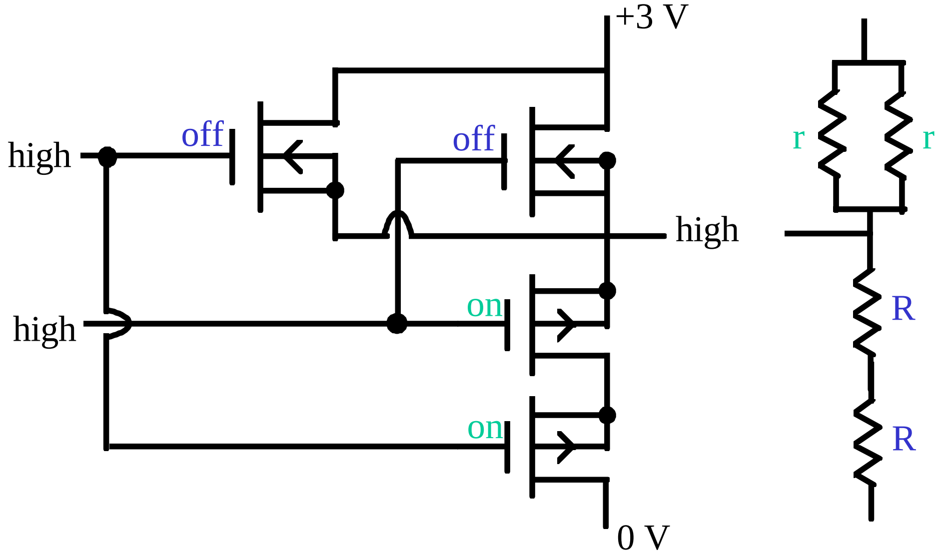
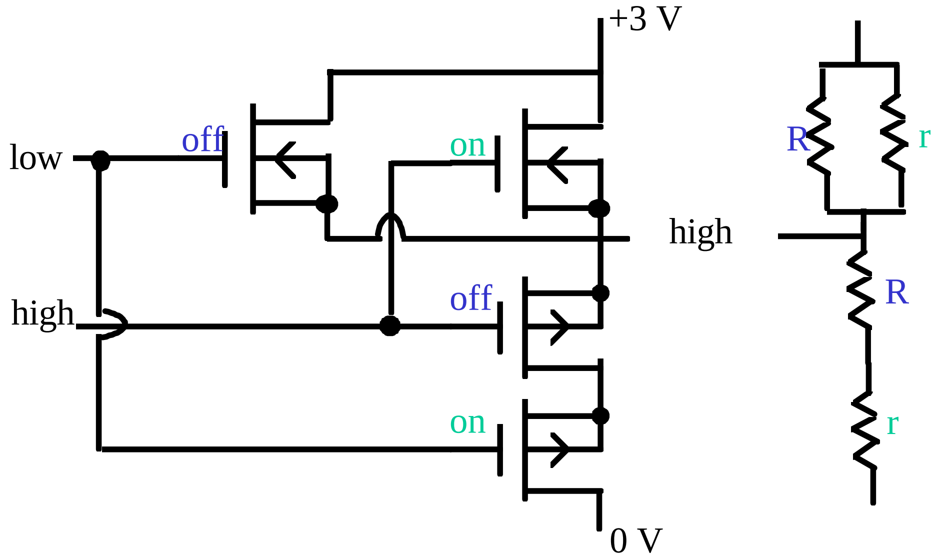


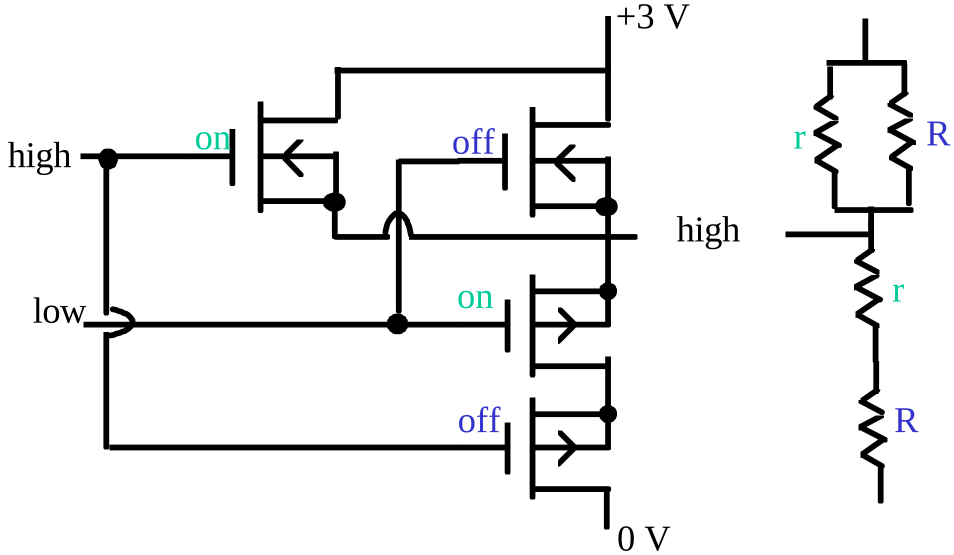
Figure 5:

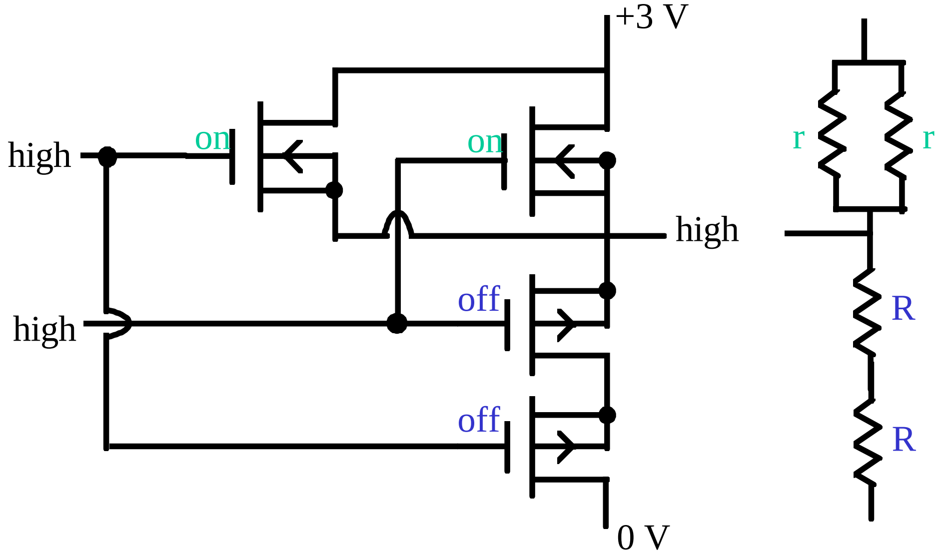


In fact, it is simply the NAND gate as drawn in *Bloomfield* with the p-channel and n-channel MOSFETs switched. Each input is connected to one p-channel and one n-channel MOSFET; therefore, for all the various possible inputs there will always be two MOSFETs “on” and two “off.” We verify that the above circuit represents an OR gate by drawing below the four possible inputs and the resulting outputs.


+3 V
high
off
off
r
r
high
high
on
R
on
R
0 V


+3 V
low
off
on
R
r
high
high
off
R
on
r
0 V


+3 V
high
on
off
r
R
high
low
on
r
off
R
0 V


+3 V
high
on
on
r
r
high
high
off
R
off
R
0 V

We draw to the side a resistive network that corresponds to the state drawn, R is a large resistance corresponding to the off state and r is a small resistance corresponding to the on state. We see that these agree with the truth table of the OR which is

Input 1	Input 2	Output
0	0	0
0	1	1
1	0	1
1	1	1

,where 1 corresponds to high voltage and 0 to low voltage.

An alternative approach in which one sends the output of a NOR gate into a NOT gate is shown below