# Physics 312 – Assignment 9

**1. Fourier integrals.** (8 points) Suppose that an EM pulse is described by the Gaussian function

$$f(t) = \frac{1}{\sqrt{2\pi\sigma^2}}\; e^{-t^2/2\sigma^2}.$$

(a) Calculate the Fourier transform $F(\omega)$ of the function $f(t)$. If you use MAPLE, remember to say `assume(sigma>0).`

(b) Define the moments of $f(t)$ and $F(\omega)$ as

$$\langle t^n \rangle = \frac{\int_{-\infty}^{\infty} t^n f(t) dt}{\int_{-\infty}^{\infty} f(t) dt}, \quad \langle \omega^n \rangle = \frac{\int_{-\infty}^{\infty} \omega^n F(\omega) d\omega}{\int_{-\infty}^{\infty} F(\omega) d\omega},$$

Calculate $\Delta t = \sqrt{\langle t^2\rangle - \langle t\rangle^2}$ and $\Delta\omega = \sqrt{\langle \omega^2\rangle - \langle \omega\rangle^2}$, and the product $\Delta t\ \Delta\omega$. What happens to the bandwidth when you make the pulse sharper? Why?

(a) The Fourier transform $F(\omega)$ of $f(t)$ is defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\; e^{-i\omega t} dt,$$

and the inverse Fourier transform of $F(\omega)$ is

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\; e^{i\omega t} d\omega.$$

(There are other definitions of the Fourier transform pairs that might switch the $e^{i\omega t}$ and $e^{-i\omega t}$ or might share the factor of $(2\pi)^{-1}$ differently.) The Fourier transform of the Gaussian function is

$$F(\omega) = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} \exp\left\{-\frac{t^2}{2\sigma^2} - i\omega t\right\}\; dt,$$

which we can rewrite as

$$F(\omega) = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}\left(t + i\sigma^2\omega\right)^2 - \frac{\sigma^2\omega^2}{2}\right\}\; dt.$$

Letting $s = t + i\sigma^2\omega$ we obtain

$$F(\omega) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{\sigma^2\omega^2}{2}\right\}\int_{-\infty}^{\infty} \exp\left\{-\frac{s^2}{2\sigma^2}\right\}\; ds.$$

Let us derive some results for Gaussian integrals, if

$$I(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda x^2}\; dx,$$

where $\lambda$ is positive; then

$$I^2(\lambda) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\lambda(x^2+y^2)} \, dx \, dy.$$

Expressing this integral in polar coordinates gives

$$I^2(\lambda) = \int_0^{\infty} \int_0^{2\pi} e^{-\lambda r^2} \, r \, dr \, d\theta.$$

The $\theta$ integration is trivial and gives a factor of $2\pi$. Let us change variables to do the remaining integral: $u = r^2$ ($rdr = du/2$)

$$I^2(\lambda) = 2\pi \int_0^{\infty} e^{-\lambda u} \, \frac{du}{2},$$

which leads to

$$I^2(\lambda) = \frac{\pi}{\lambda} \quad \text{or} \quad I(\lambda) = \sqrt{\frac{\pi}{\lambda}}.$$

Furthermore, let us define the "unnormalized moments"

$$I(\lambda, n) = \int_{-\infty}^{\infty} x^n \, e^{-\lambda x^2} \, dx.$$

We can see that $I(\lambda, n) = 0$ if $n$ is odd, since the function being integrated over is odd. Next we can find a recursion relation

$$I(\lambda, 2n+2) = -\frac{\partial}{\partial \lambda} I(\lambda, 2n),$$

provided $n \geq 0$. Thus

$$\begin{aligned} I(\lambda, 0) &= \sqrt{\pi}\lambda^{-1/2} \\ I(\lambda, 2) &= \frac{1}{2}\sqrt{\pi}\lambda^{-3/2} \\ I(\lambda, 0) &= \frac{3}{4}\sqrt{\pi}\lambda^{-5/2} \\ \text{etc.} \end{aligned}$$

Returning to the $F(\omega)$ and using the result $I(\lambda) = \sqrt{\frac{\pi}{\lambda}}$ with $\lambda = 1/2\sigma^2$, we have

$$F(\omega) = \exp\left\{-\frac{\sigma^2\omega^2}{2}\right\}.$$

(b) First of all let us state the trivial result when $n$ is odd (i.e. $n = (2m-1)$ for integer $m$)

$$\langle t^{2m-1} \rangle = \langle \omega^{2m-1} \rangle = 0.$$

We can use the results of the Gaussian integrals above to conclude that for even $n = 2m$

$$\langle t^{2m} \rangle = \left. \frac{(-1)^m \frac{\partial^m}{\partial \lambda^m} \sqrt{\frac{\pi}{\lambda}}}{\sqrt{\frac{\pi}{\lambda}}} \right|_{\lambda = 1/2\sigma^2},$$

and similarly

$$\langle \omega^{2m} \rangle = \left. \frac{(-1)^m \frac{\partial^m}{\partial \lambda^m} \sqrt{\frac{\pi}{\lambda}}}{\sqrt{\frac{\pi}{\lambda}}} \right|_{\lambda=\sigma^2/2} .$$

This leads to

$$\begin{aligned} \langle t^2 \rangle &= \sigma^2 \\ \langle t^4 \rangle &= 3\sigma^4 \\ \langle \omega^2 \rangle &= \sigma^{-2} \\ \langle \omega^4 \rangle &= 3\sigma^{-4} \\ &\text{etc.} \end{aligned}$$

$\Delta t$ is the root mean square deviation of the pulse, it measures the "spread" of the pulse. From the calculations above we conclude that $\Delta t = \sigma$ and $\Delta\omega = \sigma^{-1}$, and $\Delta t \; \Delta\omega = 1$. If we make the pulse sharper (i.e. make $\sigma$ smaller) the bandwidth $\Delta\omega$ becomes larger. Sharp features in time require a lot of frequencies in the Fourier representation. It is like the uncertainty principle, a more precise knowledge of one of the variables implies a less precise knowledge of the "complementary" variable.

**In summary:** The Fourier transform of a gaussian is a gaussian with the reciprocal width.

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**2. TV bandwidth.** (4 points) In *Melissinos*, p. 90, it is stated that the bandwidth required to transmit a television signal is $6$ MHz. Why? Try to understand this number by making an order of magnitude estimate. A television screen has $525$ horizontal lines; $30$ images are produced per second.

According to *Bloomfield,* p. 501, a typical television screen is broken into an array of dots, numbering $700 \times 525$. If each "pixel" receives a signal, and we want each to receive a signal every $1/30$ of a second, the sampling frequency $f_s$ is

$$f_s = 700 \times 525 \times 30 \text{ Hz} = 1.1 \times 10^7 \text{ Hz}.$$

According to *Melissinos*, eq. (3.20), the sampling frequency and bandwidth are related through

$$f_s = 2W,$$

a result originally due to Nyquist known as the *sampling theorem*. So $W = 5.5$ MHz.

On pp. 505-506 *Bloomfield* discusses some tricks used to squeeze information about color, sound and so forth into the $6$ MHz allotted.

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**3. Phased antenna arrays and diffraction.** (10 points) Obtain the radiation pattern shown in a "polar plot" by Melissinos in Fig. 4.4(b) and the corresponding "straight" plot of the time-averaged $dP/d$ versus angle $\theta$, as shown in the appropriate frame of the movie scatdiff.mov.

(a) You will need to generalize the equations for $E_1$ and $E_2$ at the top of page 124. Working with complex fields (to make life easier), show that for propagation at an angle $\theta$ to the $y$ axis of the figure, when $r$ tends to infinity

$$E_1 = E_0 \exp\left[i\left(kr - \frac{k\lambda}{8}\cos\theta - \omega t - \phi_1\right)\right]$$

Obtain the analogous equation for $E_2$. (Actually, $E_1$ and $E_2$ fall off like $1/r$, but this factor is cancelled when computing $dP/d$ .) Hence find $E$.

(b) The time-averaged $dP/d$ is proportional to the time-averaged $|E|^2$; the other factors do not interest us, as indicated by Melissinos. Obtain a polar plot of $\langle |E|^2 \rangle$ as found by Melissinos, as well as a "straight" plot as shown on the appropriate frame of the movie (which frame?). Accurate plots are expected.

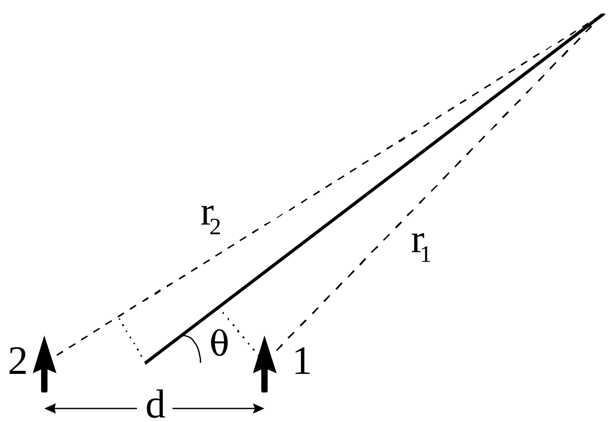


Figure 1:

The field due to the first oscillating dipole is

$$E_1 \approx E_1(r_1) \exp\left[i\left(kr_1 - \omega t - \phi_1\right)\right],$$

where $r_1$ is the distance from dipole 1 to the field point and $\phi_1$ the phase of the dipole 1's oscillation at time $t = 0$.

Similarly, the field due to the second dipole is

$$E_2 \approx E_2(r_2) \exp\left[i\left(kr_2 - \omega t - \phi_2\right)\right].$$

If we are far from the dipoles, the magnitudes are essentially equal $E_1(r_1) \approx E_2(r_2)$, so we ignore this difference. The distances $r_{1,2}$ can be related to $r$ and $d$, where $r$ is the distance from the field point to a point midway between the dipoles and where $d$ is the distance between the dipoles. One gets

$$r_{1,2} \approx r \mp \frac{d}{2}\cos\theta.$$

In this specific instance we are interested in the case $d = \lambda/4$ where $\lambda$ is the wavelength of the radiation in question. Note that $\lambda = 2\pi k$, hence, putting this altogether yields

$$E_{1,2} = E_0(r) \exp\left[i\left(k \mp \frac{\pi}{4}\cos\theta - \omega t - \phi_{1,2}\right)\right].$$

Next we want to add these two results to obtain the superposition of the two fields. Instead of working with the phases $\phi_1$ and $\phi_2$, it is more convenient to sum and the difference

$$\Phi = \phi_1 + \phi_2 \qquad \delta_\phi = \phi_1 - \phi_2$$

so that

$$\phi_1 = \frac{\Phi}{2} + \frac{\delta_\phi}{2}$$

and

$$\phi_2 = \frac{\Phi}{2} - \frac{\delta_\phi}{2}.$$

Collecting common factors

$$\begin{aligned} E_1(r) + E_2(r) &= E_0(r) \exp\left\{i(kr - \omega t - \Phi/2)\right\} \times \\ & \left[\exp\left\{-i\left[\frac{\pi}{4}\cos\theta + \frac{\delta_\phi}{2}\right]\right\} + \exp\left\{i\left[\frac{\pi}{4}\cos\theta + \frac{\delta_\phi}{2}\right]\right\}\right] \end{aligned}$$

In this particular case $\delta_\phi = \phi_1 - \phi_2 = -\pi/2$ and $|E|^2$ is proportional to

$$\cos^2[\frac{\pi}{4}\left(\cos\theta - 1\right)]$$

which is plotted below in a polar plot

and in a straight plot

This is the scattering pattern in the frame $Nx = 2,\ Ny = 1,\ \Delta x = 0.25$ of scatdiff.mov

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**4.**(4 points) In a plane electromagnetic wave, what fraction of the energy is electric and what fraction is magnetic? A precise answer is expected, and should be derived using the simple plane wave described by *Melissinos* in eqs. (4.9) and (4.9′), but the answer is true more generally.

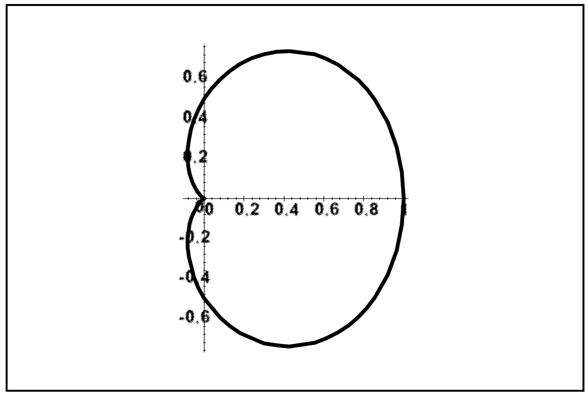


Figure 2: Polar plot

On pp. 118-119 *Melissinos* derives the wave equation from the Maxwell equations

$$\nabla^2 \mathbf{E} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

and notes that $\mu_0 \epsilon_0 = 1/c^2$ where $c$ is the speed of light. He then considers a solution of the form

$$E_x = E_0 \cos(\omega t - kz),$$

where $\omega$ is the angular frequency, $k$ the wave number and $\omega/k = c$. From the Maxwell equations he obtains the corresponding magnetic field

$$B_y = \frac{E_0}{c} \cos(\omega t - kz).$$

Next note that the (instantaneous) electric field energy density is

$$u_E = \frac{\epsilon_0}{2} E^2,$$

while the magnetic field energy density is

$$u_B = \frac{1}{2\mu_0} B^2.$$

where $E^2 = E_x^2 + E_y^2 + E_z^2$. For the solution *Melissinos* considers, the fraction of magnetic field energy density to electric field energy density is

$$\frac{u_E}{u_B} = \frac{B_y^2}{\mu_0 \epsilon_0 E_x^2} = \frac{1}{c^2 \mu_0 \epsilon_0} = 1.$$

Therefore, the ratio of magnetic field energy to electric field energy is one; or in other words, the energy density is half electric and half magnetic.

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**5. Applied diffraction.** (10 points) One important physical limitation on the resolving

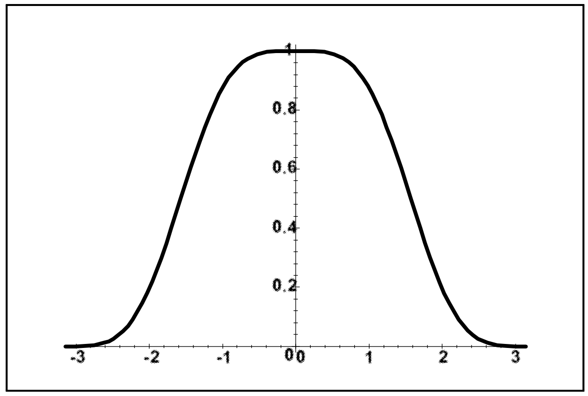


Figure 3: "Straight" plot vs $\theta$

power of an antenna is diffraction. Under ideal conditions:

(a) From how high can an eagle see a mouse on the ground?

(b) A diffraction-limited laser beam of diameter $1\ \mathrm{cm}$ is pointed at the moon. What is the diameter illuminated on the moon? Ignore atmospheric effects.

(c) The world's largest single-dish radiotelescope is at Arecibo, Puerto Rico. It has a diameter of $305\ \mathrm{m}$. What is the resolving power (the angular resolution, in degrees), when the telescope is operating at the famous $1420\ \mathrm{MHz}$ frequency of neutral hydrogen?

(d) What is the angular resolution of an array of $3000\ \mathrm{km}$, at the same frequency of $1420\ \mathrm{MHz}$?

As light passes through a slit or aperture, it diffracts. That is, the light spreads out rather than traveling in a straight line (a ray or beam). Often this phenomenon goes unnoticed because the amount of spreading or bending depends on the wavelength, and the wavelength of visible light is rather small. According to Huygens' principle each point of the slit can be thought of as a point source of light. The light from these sources interferes giving rise to diffraction patterns. Diffraction limits our ability to resolve images. If we are viewing two different sources of light through an aperture, their diffraction patterns may overlap. Rayleigh introduced a criterion saying that if the central maximum of the first source's diffraction coincides with the first minimum of the second source's diffraction pattern, the images are "just resolved." For a circular aperture this criterion leads to the condition

$$\theta_{min} = 1.22\frac{\lambda}{D},$$

where $\theta$ is the angle between the two lines that connect the sources to the center of the aperture, $\lambda$ is the wavelength of light, and $D$ is the diameter of the aperture.

(a) Let us consider the front of the mouse and the back of the mouse to be our sources and assume that a mouse has a length $\ell$. (Let's assume a mouse is $5\ \mathrm{cm}$ long.) We get an angle by dividing the $R$ distance of the mouse to the eagle's eye $\theta = \ell/R$. We need a wavelength $\lambda$ to use the formula above, we will choose $500\ \mathrm{nm}$ which is in the middle of the visible range. And we also need an aperture diameter. Let's assume an eagle's pupil is $0.2\ \mathrm{cm}$ in diameter. Then we have

$$\theta = \frac{\ell}{R} = 1.22\frac{\lambda}{D}$$

or

$$\frac{5 \times 10^{-2}\ \mathrm{m}}{R} = 1.22\frac{5 \times 10^{-7}\ \mathrm{m}}{2 \times 10^{-3}\ \mathrm{m}},$$

which yields $R \approx 160\ \mathrm{m}$ (about a tenth of a mile).

(b) The moon problem is the mouse problem in reverse, the light is coming out of the aperture instead of going into it. Plus we know $R$ the distance to the moon ($R = 3.84 \times 10^8$ m) and are looking for $\ell$. We get

$$\frac{\ell}{3.84 \times 10^8\ \mathrm{m}} = \frac{\ell}{R} = 1.22\frac{\lambda}{D} = 1.22\frac{6.32 \times 10^{-7}\ \mathrm{m}}{1 \times 10^{-2}\ \mathrm{m}},$$

where we have used $D = 1\ \mathrm{cm}$ and $\lambda = 632\ \mathrm{nm}$ (the wavelength of a Helium Neon laser). We find $\ell \approx 3 \times 10^4$ m.

(c) Here we want $\theta_{min}$, with $D = 305\ \mathrm{m}$ and $\lambda = c/f = 21\ \mathrm{cm}$. So $\theta_{min} = 8.4 \times 10^{-4}$ radians or $4.8 \times 10^{-2}$ degrees, or about 3 minutes of arc.

(d) $\theta_{min} = 8.5 \times 10^{-8}$ radians or $4.9 \times 10^{-6}$ degrees, or $0.0\,18$ seconds of arc.

**The** $21\ \mathrm{cm}$ **line.** This wavelength is rather long, much longer than wavelengths seen in the Balmer series for instance. Consequently, we are talking about a very small energy difference $\Delta E = hc/\lambda$. This small energy comes from the *hyperfine splitting*. Both the electron and the proton have spin, and associated with this spin is a magnetic dipole moment. There is a small energy corresponding to one of the magnetic dipoles being in the field caused by the other dipole which causes a slight splitting between the energies of the spin up and the spin down electron. The energy difference is $\Delta E = 5.87 \times 10^{-6}\ \mathrm{eV}$ which incidentally corresponds to a $k_B T$ with a temperature $T = 0.07^\circ K$.