Electric dipole radiation and simple antennas

The simplest radiating system is an electric dipole whose moment oscillates in time with a well-defined angular frequency ω along the z direction (for instance). We can produce such a dipole by considering two opposite charges, one of which is fixed at the origin while the other executes simple harmonic motion with amplitude d, so that its position is given by $z = d\,\cos\omega t$. To represent an atom, the moving charge would be e (considered here as a negative quantity), representing an electron, and the fixed charge would be $|e|$, representing the nuclear charge screened by the other electrons. We will use this notation for the general case as well, where e could have any value and either sign. In general

$$p_z = ez = ed\,\cos\omega t \tag{4.15}$$

and we note that (with the dot denoting differentiation with respect to t)

$$\dot{p}_z = ev = -ed\,\omega\,\sin\omega t$$

$$\ddot{p}_z = e\dot{v} = -\,ed\,\omega^2\cos\omega t$$

According to the Larmor equation (4.13) the total radiated power at time t is then

$$P = \frac{2}{3}\frac{1}{4\pi\varepsilon_0}\frac{\ddot{p}_z^2}{c^3}$$

and varies like $\cos^2\omega t$. Recalling that the time average of $\cos^2\omega t$ is $\frac{1}{2}$ and denoting the time average by brackets we find that the average radiated power is

$$\langle P\rangle = \frac{2}{3}\frac{1}{4\pi\varepsilon_0}\frac{\langle\ddot{p}_z^2\rangle}{c^3} = \frac{1}{3}\frac{e^2}{4\pi\varepsilon_0}\frac{d^2\omega^4}{c^3} \tag{4.16}$$

We have just rewritten the Larmor formula for a simple oscillator, emphasizing the difference between instantaneous power and average power. We note also that the corresponding formulas in the Gaussian system are obtained by omitting the factor $1/4\pi\varepsilon_0$.

Although the wave emitted by the oscillating dipole is a spherical wave, it does not have the same intensity in all directions. It can be shown that the average power emitted in the direction that makes an angle θ with the z axis, within a solid angle d- , is

$$\frac{d\,\langle P\rangle}{d\text{-}} = \frac{1}{8\pi}\sin^2\theta\,\frac{e^2}{4\pi\varepsilon_0}\,\frac{d^2\omega^4}{c^3} \tag{4.16'}$$

This can be understood as follows: the radiation is emitted by the component of the emitter's motion that is perpendicular to the line of sight – see Fig. 4.3(c). Note also that

$$\int\sin^2\theta\,d\text{-} \;\; = 2\pi\int_0^\pi\sin^3\theta\,d\theta = \frac{8\pi}{3}$$

so that (4.16') agrees with (4.16).

The intensity of the radiation is the energy flux, i.e. the energy dE crossing the area dA per unit time. Since energy per unit time is power, and at distance R from the source of radiation $dA = R^2 d$- , we see that the energy flux, or radiation intensity, is

$$\frac{dP}{dA} = \frac{1}{R^2}\frac{dP}{d\text{-}}$$

The intensity arriving at time t depends on the state of motion of the source at time $t - R/c$, because the signal travels with speed c. For a single dipole source in simple harmonic motion it varies like $\cos^2\omega(t - R/c)$ and its time average is found by using (4.16').

Finally, it must be mentioned that the Larmor formula is applicable only when the motion of the particle is non-relativistic, i.e. when $v << c$. For simple harmonic motion the maximum value of v is ωd, and thus (4.16) and (4.16') are strictly valid only when $\omega d << c$, or $2\pi d << \lambda$, where λ is the wavelength. This is not a serious restriction in the case of atoms, where d is about 1 ngstrom and λ is thousands of times larger, for visible light.

Antennas. Simple antennas are most efficient when they have dimensions comparable to the wavelength they emit. The theory of the simple dipole emitter is not strictly applicable to them; however it still gives qualitatively correct answers. An oscillating particle of charge e can be thought of as a current $I = I_0 \sin\omega t = 2e\omega \sin\omega t$. Replacing e by $I_0/2\omega$ and using $\omega = 2\pi c/\lambda$ as well as $1/c = (\varepsilon_0\mu_0)^{1/2}$, we can rewrite eqs. (4.16) and (4.16') in the form

$$\langle P \rangle = \frac{1}{48\pi}\left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2}\left(\frac{2\pi d}{\lambda}\right)^2 I_0^2 \tag{4.17}$$

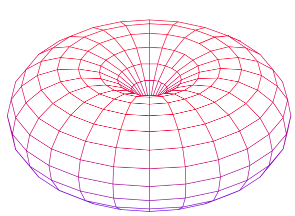
$$\frac{d\langle P \rangle}{d\text{-}} = \frac{1}{128\pi^2}\left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2}\left(\frac{2\pi d}{\lambda}\right)^2 I_0^2 \sin^2\theta \tag{4.18}$$

The quantity $Z_0 = \left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2}$ has the dimensions of impedance and is called the *impedance of free space*; it has the numerical value $Z_0 = 377$- .

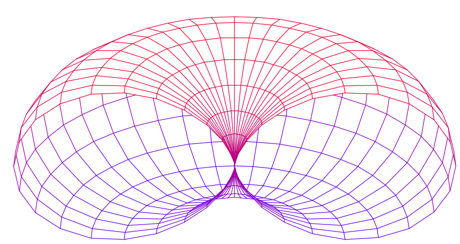
An example often used is the half-wave center-fed linear antenna – see Fig. 4.3(b). The current distribution, for $|z| \leq d/2$, is approximately of the form

$$I = I_0 \cos(\pi z/d) \sin\omega t$$

with $\omega = \pi c/d$. Since $\lambda = 2\pi c/\omega$, we see that $d = \lambda/2$, which explains the name "half-wave". The radiation pattern of the half-wave antenna is similar to the dipole pattern (4.18), and the total power is larger than the simple formula (4.17) by a factor of 1.46.



Dipole radiation pattern $\sin^2\theta$



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