Phys 312 Assignment 1 Due 23 Jan 97

1.

A parallel plate capacitor has a small hemispherical boss of radius a on one of its plates. To be definite, you can assume that A is the area of each plate, d the separation between them, with $\sqrt{A} \gg d \gg a$; take the plates to be perpendicular to the z axis and put the boss at the center of the lower plate. However, the results you will obtain hold more generally, as long as a is much smaller than the distance from the boss to the closest edge. The problem is to find the electric field.

You can imagine that the lower plate is a flat plain, the upper plate is the bottom of a large cloud, and the hemispherical boss represents a hill in the middle of the plain, or maybe a tree in the middle of a field. The problem then tells you something about the chances of being struck by lightning in hilly country, or why isolated trees are at risk of being struck.

(a) If σ_0 is the charge density on the lower plate far away from the boss, what is the electric field far away from the boss, E_0? You can (approximately) relate σ_0 to the area, A, and to the total charge on the plate, Q, but it is not necessary to do so.

(b) Find the field and the charge density everywhere on the lower plate.

(c) Where on the lower plate does the maximum electric field occur, and how much larger than E_0 is it?

(d) Where on the lower plate does the minimum electric field occur, and how much smaller than E_0 is it?

(e) Using a computer program, draw a set of equipotentials, starting from the lower plate. The graph should illustrate how the equipotentials become smooth for $z \gg a$.

Answer

(a) From Gauss Law we find $\sigma_0 = E_0\varepsilon_0$, (in SI units, where $\varepsilon_0 = 8.85\times10^{-12}\mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$) or $\sigma_0 = 4\pi E_0$ (in Gauss cgs units). See for example *Serway*, page 481, or *Tipler*, page 647. Hence $E_0 = \sigma_0/\varepsilon_0$ (SI) or $E_0 = 4\pi\sigma_0$ (Gauss cgs), is the field on the lower plate, far from the boss. It is also true that $\sigma_0 = Q/A$ + corrections of order a^2/A.

(b) This problem is equivalent to that of a conducting sphere in a uniform applied field. It is also similar, mathematically, to the problem of fluid flow past a sphere, for a non-viscous, incompressible fluid. (See last semester s notes for the very similar problem of flow past a cylinder: http://www.phys.virginia.edu/classes/311/notes/fluids11/node19.html). Anyhow, the hint was given in class to find first the electrostatic potential Φ and then use $\mathbf{E} = -\nabla\Phi$. In empty space Φ satisfies the Laplace equation $\nabla^2\Phi = 0$, and on the surface of the conducting plate Φ must be constant. Since Φ is defined up to a constant, one can set $\Phi = 0$ on the lower plate. The hint was given to look for a solution of the form

$$\Phi = A_1\ z + B_1\ z/r^3 = \left(A_1\ r + B_1/r^2\right)\cos\theta.$$

One can verify that this expression satisfies the Laplace equation expressed in spherical coordinates

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0.$$

The problem then is to determine A_1 and B_1 using the data of the problem:

(i) For large r, the field must be $E_0\hat{\mathbf{z}}$. Since Φ reduces to $A_1\ z$ for large r, we see that $A_1 = -E_0$.

(ii) Φ clearly vanishes on the plane $z = 0$. It must also vanish on the boss, i.e., for $r = a$ and $\theta \leq \pi/2$. This gives

$$-E_0\ a + B_1/a^2 = 0$$

Hence $B_1 = E_0\ a^3$ and we find

$$\Phi = -E_0\ \left(1 - \frac{a^3}{r^3}\right)\ z = -E_0\ \left(r - \frac{a^3}{r^2}\right)\ \cos\theta \tag{1}$$

The electric field is obtained from the electrostatic potential by taking a gradient: $\mathbf{E} = -\nabla\Phi$. The field on the flat part of the plate is most easily calculated in Cartesian coordinates. Recall that the gradient of a function points in the direction in which that function changes most rapidly (*Tipler*, p. 671, *Serway*, p. 716) and that Φ is constant in the plate, *i.e.* it does not vary with x or y in the flat part. Thus the gradient of Φ at the surface of the plate ($z = 0$) is in the $\hat{\mathbf{z}}$ direction, with

$$E_z = -\left.\frac{\partial\Phi}{\partial z}\right|_{z=0} = E_0\left(1 - \frac{a^3}{r^3}\right), \tag{2}$$

which is valid for $r \geq a$. The derivative is computed at constant x and y, and so the first expression in eq. (1) is convenient. (The derivatives with respect to x and y, *i.e.* the components of gradient in the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions, vanish when the expression is evaluated at $z = 0$.)

The field on the boss is most easily calculated in spherical coordinates. Again Φ is constant in the plate, that is, it does not vary with the angles θ or ϕ. Thus the gradient of Φ is in the $\hat{\mathbf{r}}$ direction, with

$$E_r = -\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 3E_0 \cos\theta, \tag{3}$$

valid for $\theta \leq \pi/2$. The derivative is computed at constant θ and ϕ, and so the second expression in eq. (1) is convenient. (The derivative with respect to ϕ is zero, and the derivative with respect to θ vanishes when the expression is evaluated at $r = a$.)

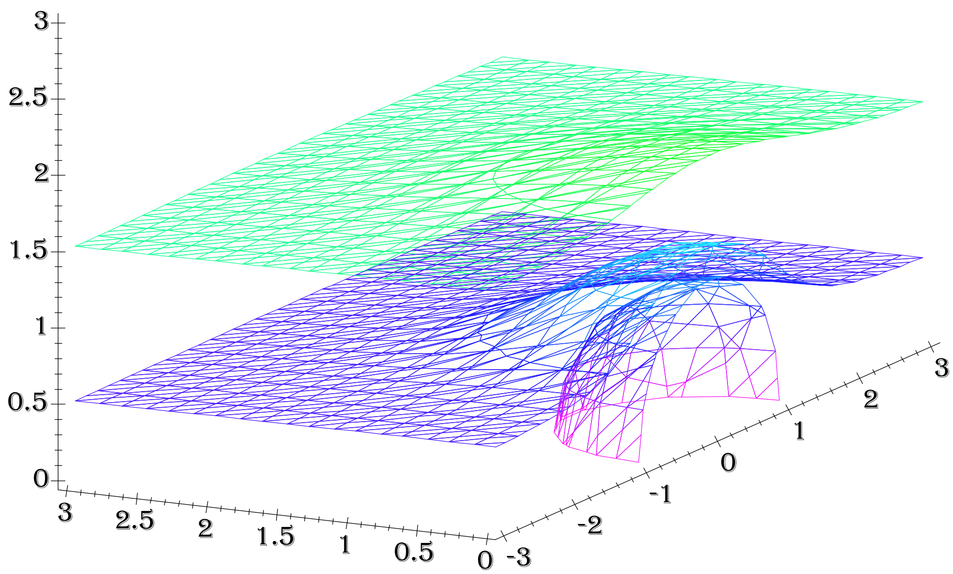
Everywhere on the plate the charge density is $E\varepsilon_0$ (in SI), or $E/4\pi$ (in Gauss cgs).

(c) Expression (2) varies from E_0 at $r \to \infty$ to 0 at $r = a$, and expression (3) varies from $3E_0$ at $\theta = 0$ to 0 at $\theta = \pi/2$. Hence, the maximum field is $3E_0$, at the top ($\theta = 0$).

(d) The minimum field is zero, at the foot of the boss. Hilltops are to be avoided during thunderstorms; it is best to be at the foot of the hill (provided the top is a safe distance away).

(e) We plot the dimensionless ratio $\Phi/E_0 a$ as a function of distance divided by a, i.e., we use $E_0 a$ as the unit of potential and a as the unit of length.

Here is a view of the boss and of the equipotentials $-\Phi = z\left(1 - \frac{1}{(x^2+y^2+z^2)^{3/2}}\right)$ for $-\Phi = 0.5$ and 1.5. where we have taken $a = 1$.



Here are plots of intersections of equipotentials with the xz plane, $-\Phi = z\left(1 - \dfrac{1}{(x^2+z^2)^{3/2}}\right)$ for $-\Phi = 0.1$ up to 0.5 (top panel) and $-\Phi = 0.5$ up to 2.5.