1.a. Given the complex number $z = 1 - i/2$, find z^2, z^3, z^4, z^5 and draw them as vectors in the complex plane. Find $1/z$, $1/z^2$, $1/z^3$, $1/z^4$, $1/z^5$ and draw them as vectors in the complex plane.

b. Given the complex number $z = \exp(i/5)$, find its first twelve powers and draw them as vectors in the complex plane. Do the same for $z = \exp(-i/5)$.

c. Find the real and imaginary parts of the complex ratio $(7+3i)/(5-4i)$. Find its magnitude (absolute value) and phase angle. Draw numerator, denominator, and ratio as vectors in the complex plane.

a. If $z = 1 - i/2$, then

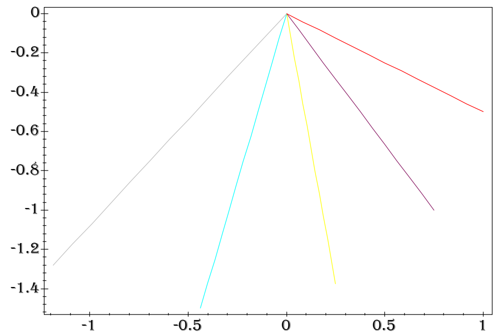
$$\begin{aligned} z^2 &= \left(1 - \frac{i}{2}\right)^2 = 1 + 2\left(-\frac{i}{2}\right) + \left(-\frac{i}{2}\right)^2 \\ &= 1 - i - \frac{1}{4} = \frac{3}{4} - i \end{aligned} \tag{3.1}$$

$$\begin{aligned} z^3 &= \left(1 - \frac{i}{2}\right)^3 = 1 + 3\left(-\frac{i}{2}\right) + 3\left(-\frac{i}{2}\right)^2 + \left(-\frac{i}{2}\right)^3 \\ &= 1 - \frac{3i}{2} - \frac{3}{4} + \frac{i}{8} = \frac{1}{4} - \frac{11i}{8} \end{aligned} \tag{3.2}$$

$$\begin{aligned} z^4 &= (z^2)^2 = \left(\frac{3}{4} - i\right)^2 = \left(\frac{3}{4}\right)^2 + 2\left(\frac{3}{4}\right)(-i) + (-i)^2 \\ &= \frac{9}{16} - \frac{3i}{2} - 1 = -\frac{7}{16} - \frac{3i}{2} \end{aligned} \tag{3.3}$$

$$\begin{aligned} z^5 &= (z^2)(z^3) = \left(\frac{3}{4} - i\right)\left(\frac{1}{4} - \frac{11i}{8}\right) \\ &= \left(\frac{3}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)\left(\frac{-11i}{8}\right) + (-i)\left(\frac{1}{4}\right) + (-i)\left(\frac{-11i}{8}\right) \\ &= \frac{3}{16} - \frac{33i}{32} - \frac{i}{4} - \frac{11}{8} = -\frac{19}{16} - \frac{41i}{32} \end{aligned} \tag{3.4}$$

The numbers z, z^2, z^3, z^4, and z^5 are plotted below.



To find an expression for $1/z$ multiply by z^*/z^*, where z^* is the *complex conjugate* of z, as follows

$$\begin{aligned} \frac{1}{z} &= \frac{1}{1-i/2} = \left(\frac{1}{1-i/2}\right)\left(\frac{1+i/2}{1+i/2}\right) \\ &= \frac{1+i/2}{1+1/4} = \frac{4}{5} + \frac{2i}{5} \end{aligned} \tag{3.5}$$

The rest, z^{-2}, z^{-3}, etc., are obtained from z^{-1}:

$$\begin{aligned} \frac{1}{z^2} &= \left(\frac{4}{5}+\frac{2i}{5}\right)^2 = \left(\frac{4}{5}\right)^2 + 2\left(\frac{4}{5}\right)\left(\frac{2i}{5}\right) + \left(\frac{2i}{5}\right)^2 \\ &= \frac{16}{25} + \frac{16i}{25} - \frac{4}{25} = \frac{12}{25} + \frac{16i}{25} \end{aligned} \tag{3.6}$$

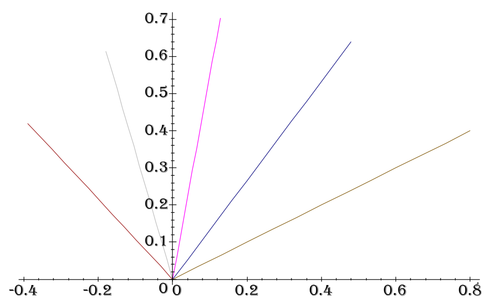
$$\begin{aligned} \frac{1}{z^3} &= \left(\frac{4}{5}+\frac{2i}{5}\right)\left(\frac{12}{25}+\frac{16i}{25}\right) \\ &= \left(\frac{4}{5}\right)\left(\frac{12}{25}\right) + \left(\frac{4}{5}\right)\left(\frac{16i}{25}\right) + \left(\frac{2i}{5}\right)\left(\frac{12}{25}\right) + \left(\frac{2i}{5}\right)\left(\frac{16i}{25}\right) \\ &= \frac{48}{125} + \frac{64i}{125} + \frac{24i}{125} - \frac{32}{125} = \frac{16}{125} + \frac{88i}{125} \end{aligned} \tag{3.7}$$

$$\begin{aligned} \frac{1}{z^4} &= \left(\frac{12}{25}+\frac{16i}{25}\right)^2 = \left(\frac{12}{25}\right)^2 + 2\left(\frac{12}{25}\right)\left(\frac{16i}{25}\right) + \left(\frac{16i}{25}\right)^2 \\ &= \frac{144}{625} + \frac{384i}{625} - \frac{256}{625} = -\frac{112}{625} + \frac{384i}{625} \end{aligned} \tag{3.8}$$

$$\frac{1}{z^5} = \left(\frac{4}{5}+\frac{2i}{5}\right)\left(-\frac{112}{625}+\frac{384i}{625}\right)$$

$$
\begin{aligned}
&= -\left(\frac{4}{5}\right)\left(\frac{112}{625}\right) + \left(\frac{4}{5}\right)\left(\frac{384i}{625}\right) + -\left(\frac{2i}{5}\right)\left(\frac{112}{625}\right) + \left(\frac{2i}{5}\right)\left(\frac{384i}{625}\right) \\
&= -\frac{448}{3125} + \frac{1536i}{3125} - \frac{224i}{3125} - \frac{768}{3125} = -\frac{1216}{3125} + \frac{1312i}{3125} \qquad (3.9)
\end{aligned}
$$

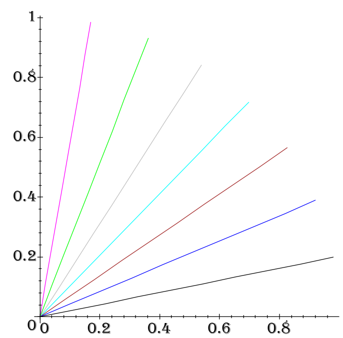
The numbers $1/z$, $1/z^2$, $1/z^3$, $1/z^4$, and $1/z^5$ are plotted below.



b. Given the form of z in this part, it is convenient to use *Euler s identity*: $\exp(i\theta) = \cos(\theta) + i\sin(\theta)$ (where θ is expressed in radians).

$$
\begin{aligned}
z &= \exp(i/5) = \cos(1/5) + i\sin(1/5) \approx 0.980 + 0.199\ i \\
z^2 &= \exp(2i/5) = \cos^2(1/5) - \sin^2(1/5) + 2i\cos(1/5)\sin(1/5) \\
&= \cos(2/5) + i\sin(2/5) \\
z^3 &= \exp(3i/5) = \cos(3/5) + i\sin(3/5) \\
z^4 &= \exp(4i/5) = \cos(4/5) + i\sin(4/5) \\
z^5 &= \exp(i) = \cos(1) + i\sin(1) \\
z^6 &= \exp(6i/5) = \cos(6/5) + i\sin(6/5) \\
z^7 &= \exp(7i/5) = \cos(7/5) + i\sin(7/5) \\
z^8 &= \exp(8i/5) = \cos(8/5) + i\sin(8/5) \\
z^9 &= \exp(9i/5) = \cos(9/5) + i\sin(9/5) \\
z^{10} &= \exp(2i) = \cos(2) + i\sin(2) \\
z^{11} &= \exp(11i/5) = \cos(11/5) + i\sin(11/5) \\
z^{12} &= \exp(12i/5) = \cos(12/5) + i\sin(12/5) \qquad (3.10)
\end{aligned}
$$

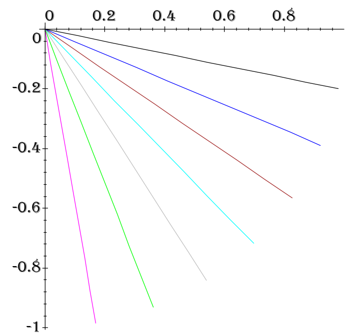
The first seven powers of z are plotted below



And for $z = \exp(-i/5)$ we find

$$
\begin{aligned}
z &= \exp(-i/5) = \cos(1/5) - i\sin(1/5) \\
z^2 &= \exp(-2i/5) = \cos^2(1/5) - \sin^2(1/5) - 2i\cos(1/5)\sin(1/5) \\
&= \cos(2/5) - i\sin(2/5) \\
z^3 &= \exp(-3i/5) = \cos(3/5) - i\sin(3/5) \\
z^4 &= \exp(-4i/5) = \cos(4/5) - i\sin(4/5) \\
z^5 &= \exp(-i) = \cos(1) - i\sin(1) \\
z^6 &= \exp(-6i/5) = \cos(6/5) - i\sin(6/5) \\
z^7 &= \exp(-7i/5) = \cos(7/5) - i\sin(7/5) \\
z^8 &= \exp(-8i/5) = \cos(8/5) - i\sin(8/5) \\
z^9 &= \exp(-9i/5) = \cos(9/5) - i\sin(9/5) \\
z^{10} &= \exp(-2i) = \cos(2) - i\sin(2) \\
z^{11} &= \exp(-11i/5) = \cos(11/5) - i\sin(11/5) \\
z^{12} &= \exp(-12i/5) = \cos(12/5) - i\sin(12/5) \qquad (3.11)
\end{aligned}
$$

The first seven of these numbers are plotted below:



c. Multiply by the *complex conjugate* of the denominator over itself to obtain

$$\begin{aligned}\frac{7+3i}{5-4i} &= \left(\frac{7+3i}{5-4i}\right)\left(\frac{5+4i}{5+4i}\right)\\ &= \frac{35+15i+28i-12}{25+16}\\ &= \frac{23+43i}{41}\end{aligned} \tag{3.12}$$

So the real part is 23/41 and the imaginary part is 43/41.

Re-express the complex number in the following way

$$\begin{aligned}re^{i\theta} &= \frac{23}{41}+\frac{43i}{41}\\ re^{i\theta} &= r\cos(\theta)+ir\sin(\theta)\end{aligned} \tag{3.13}$$

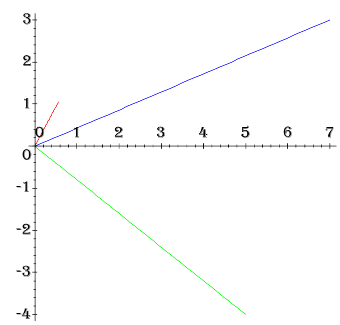
Equating real and imaginary parts gives $r\cos(\theta)=\frac{23}{41}$ and $r\sin(\theta)=\frac{43}{41}$. Squaring these and adding yields

$$\begin{aligned}r^2\cos^2(\theta)+r^2\sin^2(\theta) &= \left(\frac{23}{41}\right)^2+\left(\frac{43}{41}\right)^2\\ r^2 &= \frac{2378}{1681}=\frac{58}{41}\\ r &= \sqrt{\frac{58}{41}}\approx 1.189.\end{aligned} \tag{3.14}$$

If we take the ratio instead, we find

$$
\begin{aligned}
\frac{r\sin(\theta)}{r\cos(\theta)} &= \frac{43/41}{23/41} \\
\tan(\theta) &= \frac{43}{23} \\
\theta &= \arctan\left(\frac{43}{23}\right) \approx 1.080 \text{ radians} = 61.86 \text{ degrees.}
\end{aligned} \tag{3.15}
$$

Plotted below are $7+3i$, $5-4i$, and $(7+3i)/(5-4i)$.



---

2. Two coupled oscillators (pendulums in the class demo) obey the equations

$$m\frac{d^2x_1}{dt^2} = -m\omega_0^2 x_1 + K(x_2 - x_1)$$

$$m\frac{d^2x_2}{dt^2} = -m\omega_0^2 x_2 + K(x_1 - x_2).$$

Verify that these equations have a solution (normal mode)

$$x_1 = \cos(\omega_s t) \qquad x_2 = \cos(\omega_s t)$$

for the appropriate value of ω_s, and find ω_s. Describe in words the motion corresponding to this solution.

Similarly, verify that the equations also have a solution

$$x_1 = \cos(\omega_a t) \qquad x_2 = -\cos(\omega_a t)$$

for the appropriate value of ω_a, and find ω_a. Describe in words the motion corresponding to this solution.

The most general motion is described by a linear combination of the normal modes we have found:

$$x_1 = a\cos(\omega_s t) + b\cos(\omega_a t)$$
$$x_2 = a\cos(\omega_s t) - b\cos(\omega_a t).$$

In particular, consider the solution with $a = b$. Show that this describes the situation where, initially, pendulum 1 is held at distance $2a$ from the vertical, and released with zero speed; what are the initial position and speed of pendulum 2? Plot and describe in words the resulting motion of each pendulum when $K/m\omega_0^2 = 0.1$ (weak coupling) and $K/m\omega_0^2 = 1$ (strong coupling).

For full credit, you must give accurate plots, by MAPLE or otherwise, and they must extend over several oscillations of each pendulum.

First let us show that there is a solution to the coupled oscillator equations of the form $x_1 = \cos(\omega_s t)$ and $x_2 = \cos(\omega_s t)$. Substituting these forms into the first equations yields

$$-m\omega_s^2\cos(\omega_s t) = -m\omega_0^2\cos(\omega_s t) + K\left[\cos(\omega_s t) - \cos(\omega_s t)\right]. \tag{3.16}$$

Note that all of the terms are proportional to $\cos(\omega_s t)$ and have no other time dependence. Hence, we can divide through by $m\cos(\omega_s t)$ finding

$$\omega_s^2 = \omega_0^2. \tag{3.17}$$

Substituting the forms into the second equation yields exactly the same result. Therefore, $x_1 = x_2 = \cos(\omega_s t)$ is a solution, provided $\omega_s = \omega_0$.

In this first solution the positions of the pendulums move together, i.e. they have the same velocity and the same acceleration, and the distance between them never changes.

Now let us show that $x_1 = -x_2 = \cos(\omega_a t)$ is also a solution. Substituting these forms into the first equations yields

$$-m\omega_a^2\cos(\omega_a t) = -m\omega_0^2\cos(\omega_a t) + K\left[-\cos(\omega_a t) - \cos(\omega_a t)\right]. \tag{3.18}$$

The $\cos(\omega_a t)$ is common and can be factored out, leaving

$$\omega_a^2 = \omega_0^2 + 2\ K/m. \tag{3.19}$$

Once again, substituting these forms into the second equation yields the same result. Consequently, $x_1 = -x_2 = \cos(\omega_a t)$ is a solution, provided $\omega_a = \sqrt{\omega_0^2 + 2\ K/m}$.

In this second solution the pendulums move in opposite directions; their velocity and acceleration are in the opposite directions.

A *normal mode* is a motion of a system in which each piece moves sinusoidally with the same frequency. Thus, the solutions above are normal modes. In this particular case, the first normal mode corresponds to the *center-of-mass* motion while the second corresponds to the *relative* motion.

Next we asked to consider a solution of the form

$$x_1(t) = a\left[\cos\left(\omega_0 t\right) + \cos\left(\sqrt{\omega_0^2 + 2\ K/m}\ t\right)\right]$$

$$x_2(t) = a\left[\cos\left(\omega_0 t\right) - \cos\left(\sqrt{\omega_0^2 + 2\ K/m}\ t\right)\right],$$

where we have substituted in the expressions we found for ω_s and ω_a. The initial position of pendulum 2 is $x_2(t=0) = 0$. The velocity of pendulum 2 is $v_2(t) = dx_2(t)/dt$

$$v_2(t) = a\left[-\omega_0 \sin\left(\omega_0 t\right) + \sqrt{\omega_0^2 + 2k/m}\sin\left(\sqrt{\omega_0^2 + 2K/m}\ t\right)\right], \tag{3.20}$$

so the initial velocity of pendulum 2 is $v_2(t=0) = 0$.

Using the trigonometric identities

$$\begin{aligned} \cos A + \cos B &= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \\ \cos A - \cos B &= 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right), \end{aligned} \tag{3.21}$$

we find

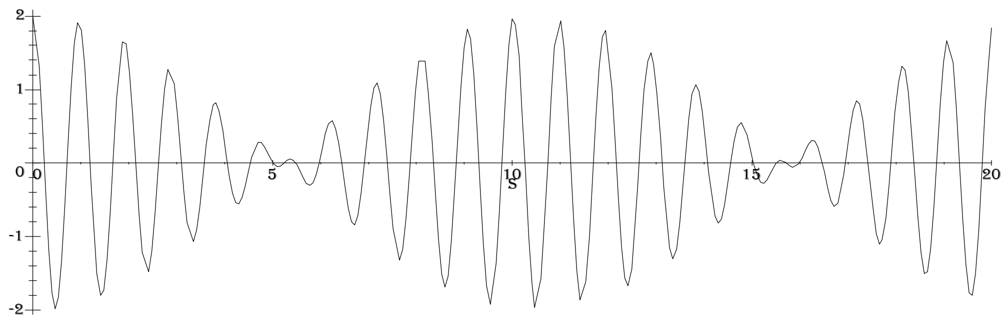
$$\begin{aligned} x_1(t) &= 2a\cos\left[\frac{\left(\omega_0 + \sqrt{\omega_0^2 + 2K/m}\right)t}{2}\right]\cos\left[\frac{\left(\omega_0 - \sqrt{\omega_0^2 + 2K/m}\right)t}{2}\right] \\ x_2(t) &= 2a\sin\left[\frac{\left(\omega_0 + \sqrt{\omega_0^2 + 2K/m}\right)t}{2}\right]\sin\left[\frac{\left(\sqrt{\omega_0^2 + 2K/m} - \omega_0\right)t}{2}\right] \end{aligned} \tag{3.22}$$

Let us express time in units of T_0 where T_0 is the natural period of the uncoupled oscillator, i.e. $T_0 = 2\pi/\omega_0$. Let $s = t/T_0$, then

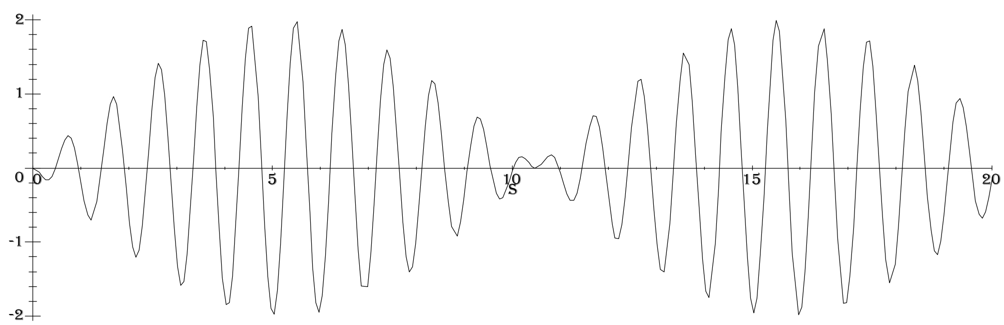
$$\begin{aligned} x_1(s) &= 2a\cos\left[\frac{2\pi\left(1+\sqrt{1+2q}\right)s}{2}\right]\cos\left[\frac{2\pi\left(1-\sqrt{1+2q}\right)2}{2}\right] \\ x_2(s) &= 2a\sin\left[\frac{2\pi\left(1+\sqrt{1+2q}\right)s}{2}\right]\sin\left[\frac{2\pi\left(\sqrt{1+2q}-1\right)s}{2}\right], \end{aligned} \tag{3.23}$$

where $q = K/m\omega_0^2$.

Below is a plot of $x_1(t) = 2\,\cos(\pi(1+\sqrt{1.2})s)\,\cos(\pi(1-\sqrt{1.2})s)$

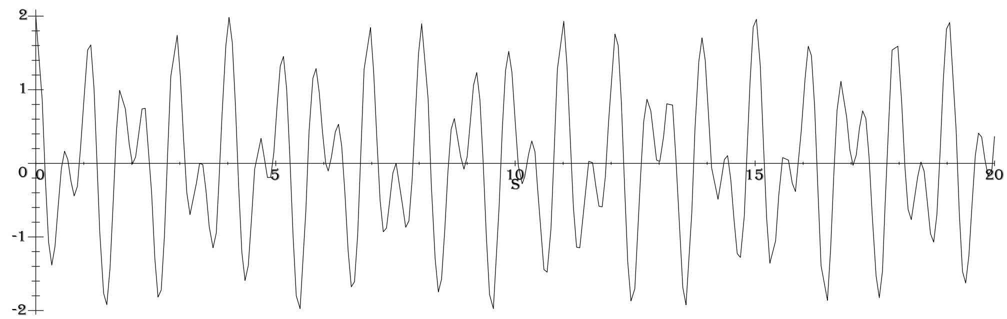


and a plot of $x_2(t) = 2\sin(\pi(1+\sqrt{1.2})s)\sin(\pi(1-\sqrt{1.2})s)$

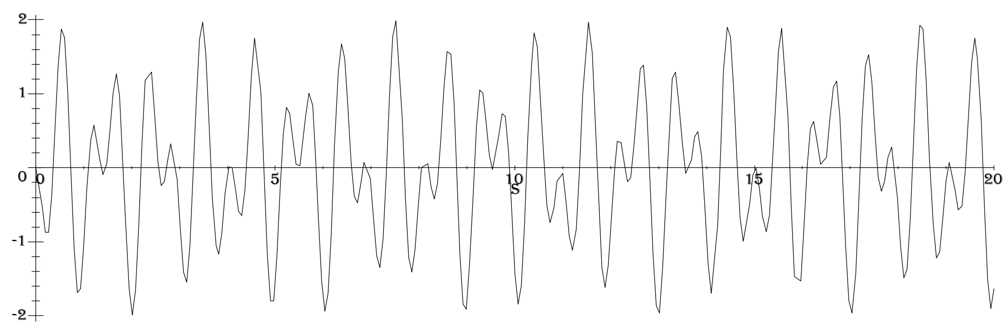


for the weak coupling case $q = 0.1$.

For the strong coupling case ($q = 1.0$), here are plots of $x_1(t) = 2\cos(\pi(1 + \sqrt{3})s)\cos(\pi(1 - \sqrt{3})s)$



and $x_2(t) = 2\sin(\pi(1 + \sqrt{3})s)\sin(\pi(1 - \sqrt{3})s)$



This should make you recall various concepts connected to waves like *interference*, *superposition*, *beats*, *phase velocity*, *group velocity*, etc.

---

3. The coupled pendulums of problem 2 behave in a way similar to two coupled quantum states. The wavefunction of two coupled quantum states is

$$\psi = c_1(t)\phi_1 + c_2(t)\phi_2$$

and the time evolution is given by

$$i\hbar\frac{dc_1}{dt} = E_1c_1 + Vc_2$$

$$i\hbar\frac{dc_2}{dt} = E_2c_2 + Vc_1.$$

Consider only the degenerate" case when $E_1 = E_2$ (identical atoms, for instance), analogously to the identical pendulums of problem 2. Use E_0 to denote the common value of E_1 and E_2.

Verify that these equations have the solution (eigenstate)

$$c_1 = \exp(-i\omega_s t) \qquad c_2 = \exp(-i\omega_s t)$$

for the appropriate value of ω_s and find ω_s. As a function of time, what is the probability that the quantum system is in the state ϕ_1 (if it is in this eigenstate)? In the state ϕ_2?

Similarly, verify that the equations have the solution

$$c_1 = \exp(-i\omega_a t) \qquad c_2 = -\exp(-i\omega_a t)$$

for the appropriate value of ω_a, and find ω_a. As a function of time, what is the probability that the quantum system is in the state ϕ_1? In state ϕ_2?

The most general solution is described by a linear combination of the eigenstates we have found:

$$c_1 = a\exp(-i\omega_s t) + b\exp(-i\omega_a t)$$

$$c_2 = a\exp(-i\omega_s t) - b\exp(-i\omega_a t).$$

In particular, consider the solution with $a = b$. Show that this describes the situation where, initially, the system is in the state ϕ_1. Plot (accurately) and describe in words the probabilities $|c_1|^2$ and $|c_2|^2$, as they vary with time, when $V/E_0 = 0.1$ (weak coupling) and $V/E_0 = 1.0$ (strong coupling).

Let us substitute the proposed solution $c_1 = c_2 = \exp(-i\omega_s t)$ into the time evolution equations

$$\hbar\omega_s \exp(-i\omega_s t) = E_0 \exp(-i\omega_s t) + V\exp(-i\omega_s t). \tag{3.24}$$

Note that $\exp(-i\omega_s t)$ is common to all of the terms and constitutes the full time dependence of each. We can divide through by $\exp(-i\omega_s t)$ which gives us

$$\hbar\omega_s = E_0 + V. \tag{3.25}$$

Therefore, $c_1 = c_2 = \exp(-i\omega_s t)$ is a solution provided $\omega_s = (E_0 + V)/\hbar$. (From the units of $\hbar$ and ω we can determine that E_0 and V are energies.)

When we write the expression for a wavefunction in terms of states

$$\psi = c_1(t)\phi_1 + c_2(t)\phi_2,$$

it is somewhat analogous to writing a position vector in terms of its components $\mathbf{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath}$. In particular, we are assuming here that ϕ_1 is *normalized* (somewhat like $\hat{\imath}\cdot\hat{\imath} = 1$) and furthermore that ϕ_1 and ϕ_2 are *orthogonal* (somewhat like $\hat{\imath}\cdot\hat{\jmath} = 0$). In this analogy the c s are amplitudes in the directions ψ_1 and ψ_2; and they are often called *probability amplitudes*. If we are in the state ψ with $c_1 = c_2 = \exp[-i(E_0 + V)t/\hbar]$, then the probability that the quantum system is in the state ϕ_1 at time t is proportional to $|c_1(t)|^2$. The sum of all the distinct probabilities must add up to one, so in this case, the probability to be in state ϕ_1 is

$$P_{\phi_1} = \frac{|c_1(t)|^2}{|c_1(t)|^2 + |c_2(t)|^2} = \frac{1}{2}. \tag{3.26}$$

And similarly the probability to be in state ϕ_2 is

$$P_{\phi_2} = \frac{|c_2(t)|^2}{|c_1(t)|^2 + |c_2(t)|^2} = \frac{1}{2}. \tag{3.27}$$

Note that we have divided by $|c_1(t)|^2 + |c_2(t)|^2$ so that the probabilities add up to one; this procedure is called *normalization*.

Now we consider another solution. Putting $c_1 = -c_2 = \exp(-i\omega_a t)$ into the evolution equation yields

$$\hbar\omega_a \exp(-i\omega_a t) = E_0 \exp(-i\omega_a t) - V \exp(-i\omega_a t), \tag{3.28}$$

leading to $\omega_a = (E_0 - V)/\hbar$. The probability that the quantum system is in the state ϕ_1 is then given by

$$P_{\phi_1} = \frac{|c_1(t)|^2}{|c_1(t)|^2 + |c_2(t)|^2} = \frac{1}{2}. \tag{3.29}$$

And the probability to be in state ϕ_2 is

$$P_{\phi_2} = \frac{|c_2(t)|^2}{|c_1(t)|^2 + |c_2(t)|^2} = \frac{1}{2}. \tag{3.30}$$

The subscripts s and a were not chosen arbitrarily; s stands for symmetric" and a for antisymmetric." Note that if we were to exchange" the labels $1 \leftrightarrow 2$ that $\psi \leftrightarrow \psi$ in the symmetric case and $\psi \leftrightarrow -\psi$ in the antisymmetric case.

Now we consider the linear combination of the eigenstates with the following coefficients

$$\begin{aligned} c_1 &= a\left[\exp\left(-i\frac{(E_0+V)t}{\hbar}\right) + \exp\left(-i\frac{(E_0-V)t}{\hbar}\right)\right] \\ c_2 &= a\left[\exp\left(-i\frac{(E_0+V)t}{\hbar}\right) - \exp\left(-i\frac{(E_0-V)t}{\hbar}\right)\right]. \end{aligned} \tag{3.31}$$

The probability that the quantum system is in the state ϕ_1 is given by

$$P_{\phi_1} = \frac{|c_1(t)|^2}{|c_1(t)|^2 + |c_2(t)|^2}, \tag{3.32}$$

where

$$\begin{aligned} |c_1(t)|^2 &= |a|^2\left[2 + 2\cos\left(\frac{2Vt}{\hbar}\right)\right] \\ |c_2(t)|^2 &= |a|^2\left[2 - 2\cos\left(\frac{2Vt}{\hbar}\right)\right] \end{aligned} \tag{3.33}$$

Note that $|c_1(t)|^2 + |c_2(t)|^2 = |a|^2$ at all times. The probability to be in state ϕ_1 is

$$P_{\phi_1}(t) = \frac{1 + \cos(2Vt/\hbar)}{2}; \tag{3.34}$$

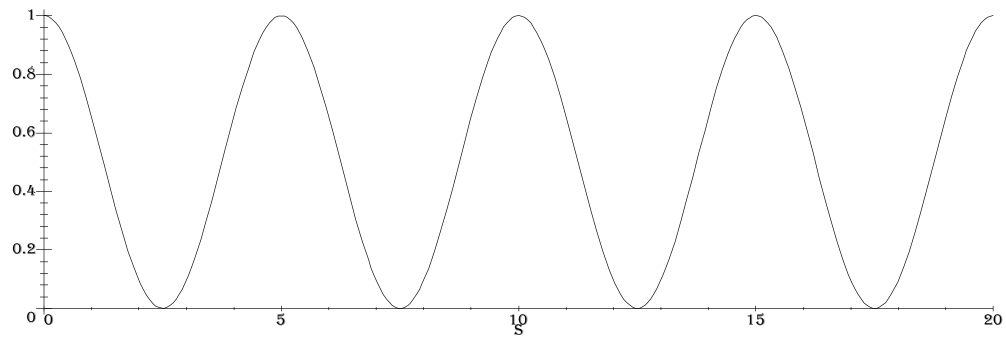
and that to be in state ϕ_2 is

$$P_{\phi_2}(t) = \frac{1 - \cos(2Vt/\hbar)}{2}. \tag{3.35}$$

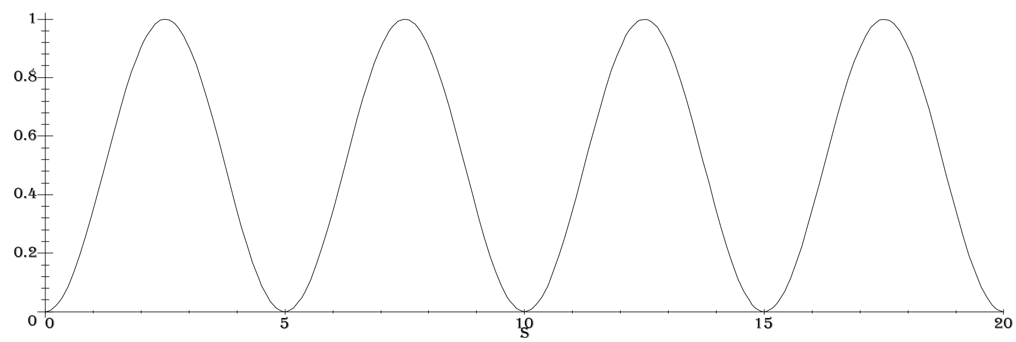
It is the V term which couples the two states ϕ_1 and ϕ_2. The period in the absence of any coupling is $T_0 = 2\pi\hbar/E_0$, so let us express t in terms of T_0, i.e. $s = t/T_0$. The probabilities are then

$$\begin{aligned} P_{\phi_1}(s) &= \frac{1 + \cos[4\pi s(V/E_0)]}{2} \\ P_{\phi_2}(s) &= \frac{1 - \cos[4\pi s(V/E_0)]}{2}. \end{aligned} \tag{3.36}$$

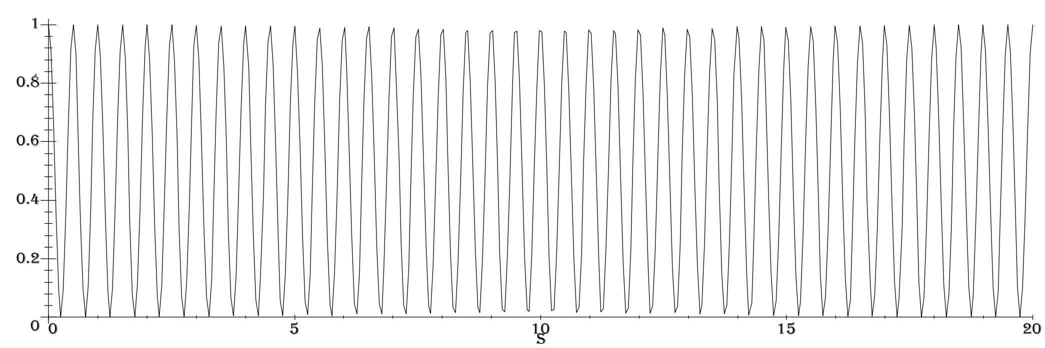
For weak coupling ($V/E_0 = 0.1$) we have $P_1(s) = (1 + \cos(0.4\pi s))/2$



and $P_2(s) = (1 - \cos(0.4\pi s))/2$



And for the strong coupling case $(V/E_0 = 1.0)$ $P_1(s) = (1 + \cos(4\pi s))/2$



and $P_2(s) = (1 - \cos(4\pi s))/2$