Passive low pass filter: As I mentioned in class, one stage of an archaic AM receiver consists of a low pass filter which removes the RF (radio frequency) carrier wave, leaving the AF (audio frequency) signal. The simplest passive filter (i.e., a filter which does not involve active elements such as transistors), consists of a resistor and a capacitor.

(a) Sketch this filter.
(b) Show that
\[
\left| \frac{V_{\text{out}}}{V_{\text{in}}} \right|^2 = \frac{1}{1 + (\omega RC)^2}.
\]
Plot this transfer function and show that it has the right properties to function as a low pass filter.
(c) Choose appropriate values of \( R \) and \( C \) for an AM receiver.
(d) Our simple filter has a very slow “roll-off”; ideally, one would like a “brick wall” transfer function which is 1 up to some frequency \( \omega_0 \) and zero otherwise. Some improvement is obtained by incorporating an inductor into the filter, as discussed in class. Sketch this filter.
(e) Find the transfer function for this filter. Show that the response at low frequencies is the flattest when \( R = \sqrt{2L/C} \). This can be done graphically by plotting the transfer function as a function of a dimensionless frequency for several values of a dimensionless parameter that is proportional to \( L \). Note that for \( L = 0 \), we are back to the previous case.
(f) How does the transfer function behave at high frequencies?
(a) Below is a sketch of a simple low pass filter.

![Low Pass Filter Diagram]

(b) In Problem 2, we wrote down the solution for the charge on the capacitor in an LRC circuit. To get $V_{out}$, the voltage across the capacitor, we simply divide by $C$.

$$V_{out} = \frac{E_{max}e^{-i\omega t}}{LC \left( \frac{1}{j\omega} - \omega^2 \right) - iRC \omega}$$

where we have dropped the transient part. The transfer function is

$$\left| \frac{V_{out}}{V_{in}} \right|^2 = \frac{\langle (\text{Re } V_{out})^2 \rangle_T}{\langle (\text{Re } V_{in})^2 \rangle_T},$$

if we use the complex notation for the $V$’s. This is the same as

where $\langle \rangle_T$ refers to the average over a period of the ac source. If we used the real representation of $V$, we would have to use the latter expression.

The transfer function for the LRC circuit is then

$$\left| \frac{V_{out}}{V_{in}} \right|^2 = \frac{1}{\left(1 - LC\omega^2\right)^2 + R^2C^2\omega^2}$$

In the case where there is no inductor ($L = 0$), it becomes

$$\left| \frac{V_{out}}{V_{in}} \right|^2 = \frac{1}{1 + R^2C^2\omega^2}.$$
where the $x$ axis is measured in units of $1/RC$, i.e. $w = \omega RC$.

(c) For an AM radio, the desired signal should have frequencies between 30 Hz and 20,000 Hz which correspond to frequencies we can hear (Bloomfield, p. 347), and the carrier frequency is between 550 kHz and 1000 kHz, which are the AM radio frequencies (Bloomfield, p. 496). Thus we want our transfer function to be close to one for audible frequencies but to fall to zero for the AM radio frequencies. So if we choose

$$RC = \frac{1}{20,000 \text{ Hz}},$$

then the transfer function is 0.9753 at the highest audible frequency and 0.0496 at the lowest AM radio frequency.
And as already calculated above the transfer function is
\[
\left| \frac{V_{\text{out}}}{V_{\text{in}}} \right|^2 = \frac{1}{(1 - LC\omega^2)^2 + R^2C^2\omega^2}
\]

Let us use again the scaled frequency \( w = RC\omega \) then
\[
\left| \frac{V_{\text{out}}}{V_{\text{in}}} \right|^2 = \frac{1}{(1 - \frac{L}{RC}w^2)^2 + w^2}
\]

Note the presence of the dimensionless quantity \( L/R^2C \) which we identified in Problem 2 with \( 1/Q^2 \), where \( Q \) is the quality factor of the LRC circuit. Then
\[
\left| \frac{V_{\text{out}}}{V_{\text{in}}} \right|^2 = \frac{1}{(1 - w^2/Q^2)^2 + w^2}
\]

The desire in a low pass filter is to have the transfer function as flat as possible for the small frequencies and to fall off as rapidly as possible for the undesired high frequencies. Adding the inductor can improve our low pass filter on both scores.

Because the transfer function is even its first derivative is zero at \( w = 0 \), but we can choose \( Q \) to make the second derivative at \( w = 0 \) zero as well.
\[
\frac{\partial^2}{\partial w^2} \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right|^2 \bigg|_{w=0} = -2 + \frac{4}{Q^2}
\]

So choosing \( 1/Q^2 = L/R^2C = 1/2 \) makes the transfer function flat at small frequencies.
Let’s look at it graphically as well. At $Q = 2$, $1/Q^2 = 0.25$, the $LRC$ transfer function looks like

At $1/Q^2 = 0.5$

At $1/Q^2 = 0.75$
(f) At large frequencies, the denominator is dominated by the $\omega^4$ term, so that the transfer function varies like $\omega^{-4}$ for large $\omega$. 