Assignment 9

Problem 3

**Phased antenna arrays and diffraction.** (10 points) Obtain the radiation pattern shown in a “polar plot” by Melissinos in Fig. 4.4(b) and the corresponding “straight” plot of the time-averaged $dP/d\theta$ versus angle $\theta$, as shown in the appropriate frame of the movie scat-diff.mov.

(a) You will need to generalize the equations for $E_1$ and $E_2$ at the top of page 124. Working with complex fields (to make life easier), show that for propagation at an angle $\theta$ to the $y$ axis of the figure, when $r$ tends to infinity

$$E_1 = E_0 \exp \left( i \left( kr \frac{k\lambda}{8} \cos \theta - \omega t - \phi_1 \right) \right)$$

Obtain the analogous equation for $E_2$. (Actually, $E_1$ and $E_2$ fall off like $1/r$, but this factor is cancelled when computing $dP/d\theta$.) Hence find $E$.

(b) The time-averaged $dP/d\theta$ is proportional to the time-averaged $|E|^2$; the other factors do not interest us, as indicated by Melissinos. Obtain a polar plot of $\langle |E|^2 \rangle$ as found by Melissinos, as well as a “straight” plot as shown on the appropriate frame of the movie (which frame?). Accurate plots are expected.

The field due to the first oscillating dipole is

$$E_1 \approx E_1(r_1) \exp \left[i \left( kr_1 - \omega t - \phi_1 \right) \right],$$

where $r_1$ is the distance from dipole 1 to the field point and $\phi_1$ the phase of the dipole 1’s
oscillation at time $t = 0$.

Similarly, the field due to the second dipole is

$$E_2 \approx E_2(r_2) \exp \left[ i (kr_2 - \omega t - \phi_2) \right].$$

If we are far from the dipoles, the magnitudes are essentially equal $E_1(r_1) \approx E_2(r_2)$, so we ignore this difference. The distances $r_{1,2}$ can be related to $r$ and $d$, where $r$ is the distance from the field point to a point midway between the dipoles and where $d$ is the distance between the dipoles. One gets

$$r_{1,2} \approx r \mp \frac{d}{2} \cos \theta.$$

In this specific instance we are interested in the case $d = \lambda/4$ where $\lambda$ is the wavelength of the radiation in question. Note that $\lambda = 2\pi k$, hence, putting this altogether yields

$$E_{1,2} = E_0(r) \exp \left[ i \left( k \mp \frac{\pi}{4} \cos \theta - \omega t - \phi_{1,2} \right) \right].$$

Next we want to add these two results to obtain the superposition of the two fields. Instead of working with the phases $\phi_1$ and $\phi_2$, it is more convenient to sum and the difference

$$\Phi = \phi_1 + \phi_2 \quad \delta_\phi = \phi_1 - \phi_2$$

so that

$$\phi_1 = \frac{\Phi}{2} + \frac{\delta_\phi}{2}$$

and

$$\phi_2 = \frac{\Phi}{2} - \frac{\delta_\phi}{2}.$$ Collecting common factors

$$E_1(r) + E_2(r) = E_0(r) \exp \left\{ i(kr - \omega t - \Phi/2) \right\} \times \left[ \exp \left\{ -i \left( \frac{\pi}{4} \cos \theta + \frac{\delta_\phi}{2} \right) \right\} + \exp \left\{ i \left( \frac{\pi}{4} \cos \theta + \frac{\delta_\phi}{2} \right) \right\} \right\}$$

In this particular case $\delta_\phi = \phi_1 - \phi_2 = -\pi/2$ and $|E|^2$ is proportional to

$$\cos^2 \left( \frac{\pi}{4} \cos \theta - 1 \right)$$

which is plotted below in a polar plot.
Polar plot

and in a straight plot

This is the scattering pattern in the frame $N_x = 2$, $N_y = 1$, $\Delta x = 0.25$ of scatdiff.mov