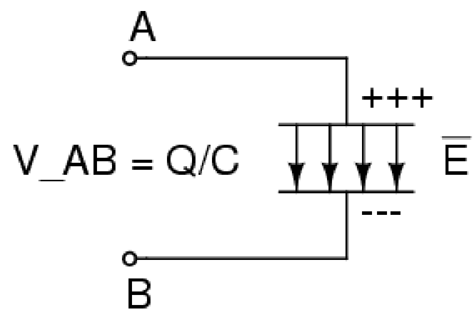
The Capacitor

Two conductors in close proximity (and electrically isolated from one another) form a capacitor. An electric field is produced by charge differences between the conductors.

The capacitance of such a device is defined by:
C [Farads] = V [Volts] / Q [Coulombs], where Q is the differential charge between the conductors.
i.e. A capacitor of 1 Farad in capacity with 1 Coulomb of stored charge will have 1 Volt of potential difference across its leads.



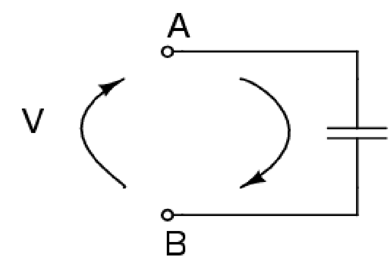
i) Charges on the plates have equal magnitude and opposite sign.
ii) Positive Q at A give V_AB > 0

An approximation of the capacitance formed by two conductors (ignoring edge effects) is given by:

$$C = \frac{A\varepsilon}{d}$$

$$A = overlap\ area\ between\ plates\,[m^2]$$
$$\varepsilon = dielectric\ permittivity\ of\ media\ between\ the\ plates\,[F\,m^{-1}]$$
$$d = distance\ betwen\ the\ plates\,[m]$$

A capacitor can pass a time varying signal, but is an open circuit for DC. A surge of charge onto one plate causes an identical surge of charge out of the other plate.

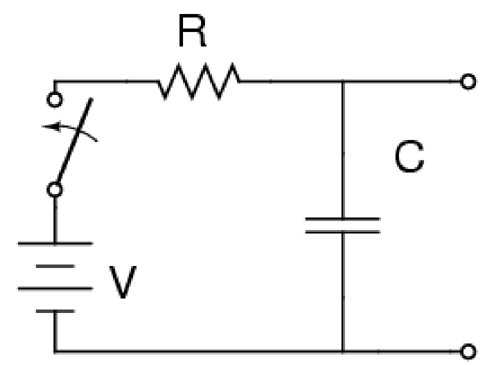


differentiating V = Q/C

$$\frac{dV}{dt} = \frac{1}{C}\frac{dQ}{dt} = \frac{I}{C} \rightarrow current \propto \frac{dV}{dt}$$

A simple RC Circuit

A constant voltage source applied at t=0 by closing a switch. Using KVL:



$$V = IR + \frac{1}{C}\int I\,dt$$

differentiating each term yields:

$$\frac{dI}{dt} + \frac{1}{RC}I = 0$$

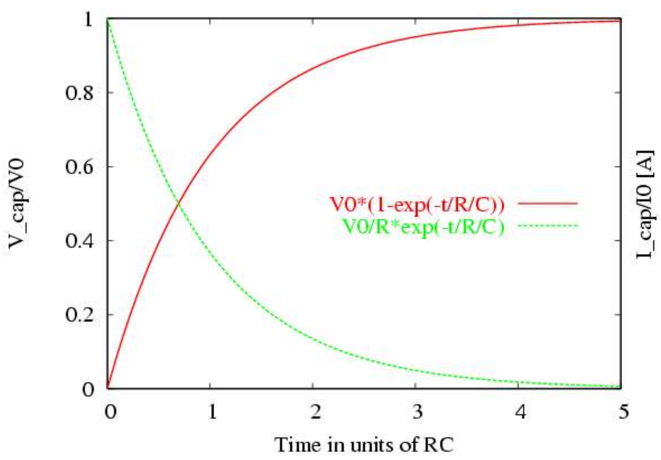
solving for the current gives $I = I_0 e^{-t/RC}$

$$V_C = \frac{1}{C}\int_0^t I\,dt = V_0(1-e^{-t/RC})$$

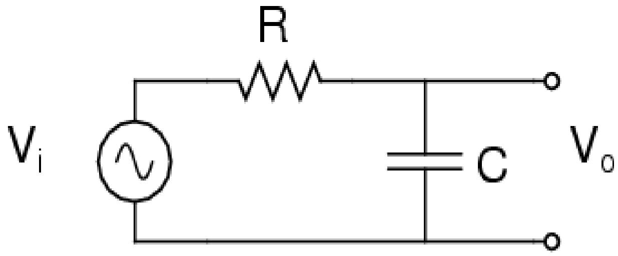
$$I_C = C\frac{dV}{dt} = \frac{1}{R}e^{-t/RC}$$

Voltage across R is given by:

$$V_R = V_0 e^{-t/RC}$$



<u>RC integrator / Low pass filter</u>



$$V_O = \frac{Q}{C} = \frac{\int I\,dt}{C}$$

KVL:

$$V_i - IR - V_o = 0$$

$$subs.\quad I = C\frac{dV}{dt}$$

$$\frac{dV_o}{dt} + \frac{1}{RC}V_o = \frac{1}{RC}V_i$$

Note: if $V_o \ll V_i \rightarrow \frac{dV_o}{dt} \simeq \frac{1}{RC}V_i$

and $V_o \simeq \frac{1}{RC}\int_0^t V_i(t')\,dt'$

In other words this circuit can approximate the integral of V_i

The more rigorous solution:

Multiply each side of the differential equation by the integration factor: $e^{(t/RC)}$

$$\frac{d}{dt}(V_o(t)e^{t/RC}) = \frac{V_i(t)}{RC}e^{t/RC}$$

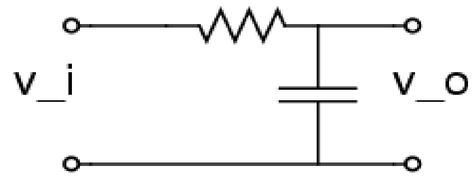
$V_o(t) = \frac{1}{RC}\int_0^t V_i(t')e^{-(t-t')/RC}\,dt' + \cancel{V_o(0)}e^{-t/RC}$ take $V_o(0)=0$ it's just a potential...

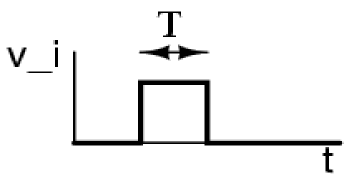
now consider the limiting case:

$$V_o(t) \simeq \int_0^t V_i(t')\,dt' \quad \text{for} \quad t \ll RC$$

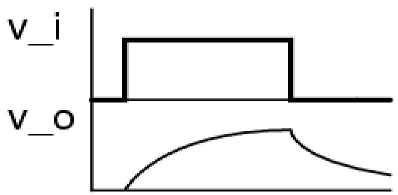
For times small compared to RC the circuit integrates. This is also the region in which V_o << V_i.

Integrator response to square pulse

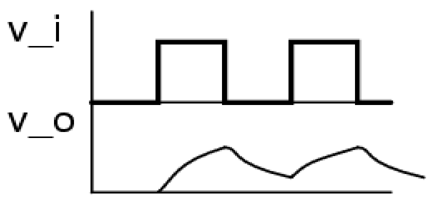




define:
T ≡ pulse width
$\tau \equiv RC$

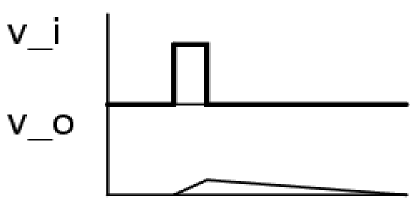


$\tau \ll$ T   Bad integrator



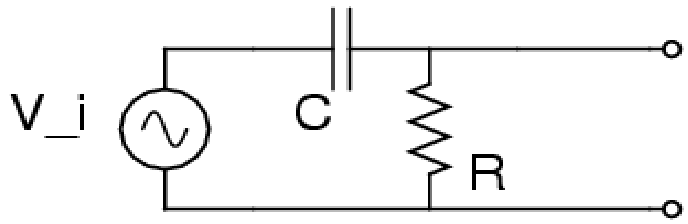
$\tau \approx$ T

Qualitatively: The capacitor “rounds” sharp corners of V_i. Later we'll see that high frequency components are approximately shorted to ground in this configuration.



$\tau \gg$ T   Good integrator
But V_o is very small

RC differentiator / high pass filter / DC blocker



$V_O = IR$

KVL:

$$V_i - \frac{Q}{C} - V_o = 0$$

Differentiate and use $I = \frac{V_o}{R}$

$$\frac{dV_o}{dt} + \frac{1}{RC} V_o = \frac{dV_i}{dt}$$

Note: if $\frac{dV_o}{dt} \ll \frac{dV_i}{dt} \to V_o(t) \simeq RC \frac{dV_i}{dt}$

In this approximation, the circuit differentiates V_i

Rigorous solution obtained again by multiplying the integration factor

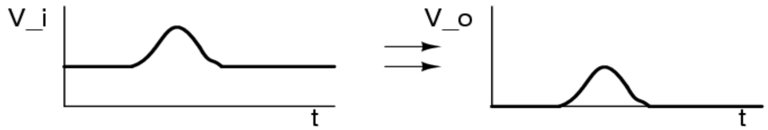
Multiply each side of the differential equation by the integration factor: $e^{(t/RC)}$

$$\frac{d}{dt}(V_o(t) e^{t/RC}) = \frac{dV_i(t)}{dt} e^{t/RC}$$

$V_o(t) = \int_0^t \frac{dV_i(t')}{dt'} e^{-(t-t')/RC} dt' + \cancel{V_o(0)} e^{-t/RC}$ take $V_o(0) = 0$ it's just a potential...

Consider two limiting cases:

(1)t << RC “DC blocking regime”



use $e^{-(t-t')/RC} \to 1$ for $0 < t' < t$

then $V_o(t) = \int \frac{dV_i(t')}{dt'} dt' = V_i(t) - V_i(0)$

V_o follows changes in V_i

This is useful in situations where the time varying signal voltage is sitting on a DC pedestal voltage.

The capacitor “blocks” the DC component.

rewrite $V_o(t) = \int_0^t \frac{dV_i(t')}{dt'} e^{-(t-t')/RC} dt'$ using partial integration $\int_a^b u\, dv = [uv]_a^b - \int_a^b v\, du$

(2)t>>RC Differentiator

$$V_o(t) = \left[V_i(t) - \cancel{V_i(0) e^{-t/RC}}\right] - \frac{1}{RC} \int_0^t V_i(t') e^{-(t-t')/RC} dt'$$

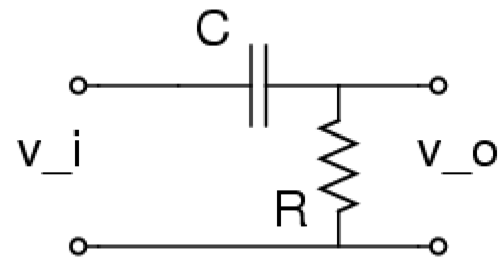
for t>>RC the integral is dominated for values t' ~ t

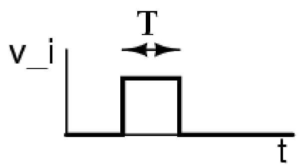
Approximation for V_i is valid: $V_i(t') \simeq V_i(t) - \frac{d}{dt} V_i(t)(t-t')$ and

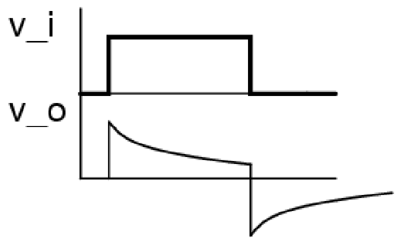
$$V_o(t) \simeq V_i(t) - \underbrace{V_i(t) \int_0^t e^{-(t-t')/RC} \frac{dt'}{RC}}_{\to 1\, as \frac{t}{RC} \to \infty} + \underbrace{RC \frac{dV_i(t)}{dt} \int_0^t (t-t') e^{-(t-t')/RC} \frac{dt'}{(RC)^2}}_{\to 1\, as \frac{t}{RC} \to \infty}$$ In this regime, the circuit differentiates.

thus: $V_o(t) \simeq RC \frac{dV_i(t)}{dt}$

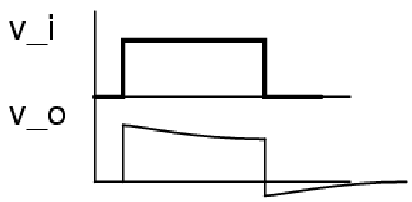
Differentiator response to square pulse



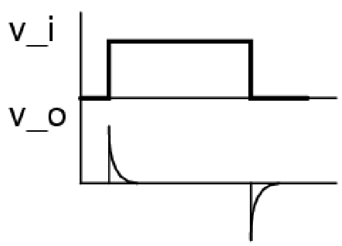




$\tau \sim T$ Bad differentiator

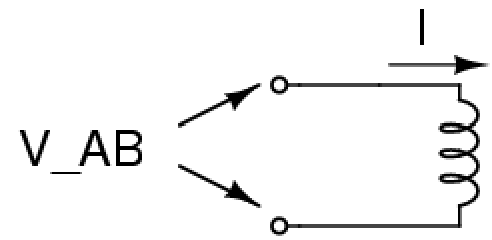


$\tau \gg T \sim$ D.C. Blocker
output sags at flat-top,
also slight overshoot on trailing edge



$\tau \ll T$ Good Differentiator
but V_o is small

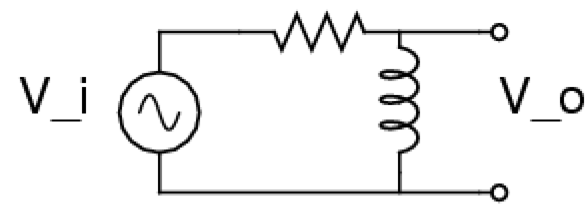
Inductors:



V_AB = L dI/dt
Lenz's law gives sign of V and dI/dt. If V_AB>0 then I is increasing in the direction of the arrow.

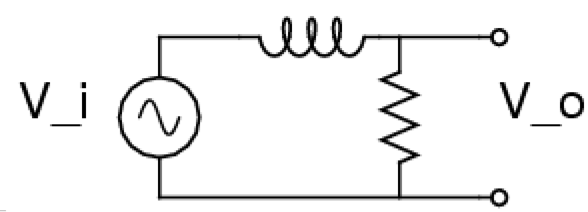
Integrators/differentiators may be constructed with LR circuits, similar to the RC examples we have seen, but there is no equivalent of the DC blocker.

High Pass Configuration



(high frequencies imply large dI/dt, thus large voltage drop across inductor. This will become more clear when we discuss complex impedance.)

Low Pass Configuration

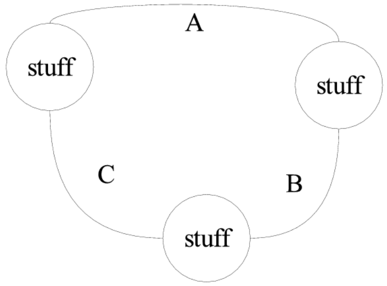


Previous equations derived for RC circuits also apply here, but RC → L/R = τ

Here we have reached a severe disadvantage of time-domain analysis. The relationship between V and I for a capacitor or an inductor is <u>not</u> linear, in the restricted sense like Ohm's Law. Applying network analysis techniques to a network containing L/C components yields a matrix of differential equations instead of a simple linear system.

We regain linearity in the network analysis by looking instead at the frequency dependence of the circuit.

Frequency Domain:



KVL, KCL and Ohm's Laws hold for each instant of time.

For example: $V_{AB}(t)+V_{BC}(t)+V_{CA}(t)=0$

This can be expressed in terms of the Fourier Transform of the voltages as follows

$$V_{AB}(t)+V_{BC}(t)+V_{CA}(t)=0=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[\tilde{V}_{AB}(\omega)+\tilde{V}_{BC}(\omega)+\tilde{V}_{CA}(\omega)]e^{j\omega t}d\omega$$

where $j\equiv\sqrt{-1}$

also $e^{j\omega t}=\cos(\omega t)+j\sin(\omega t)$

For electronics applications *j* is used instead of *i* to prevent confusion with notations for current.

The completeness relation $\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{(\omega-\omega')t}dt=\delta(\omega-\omega')\}$ allows us to restate the KVL in terms of the Fourier amplitudes:

$$\tilde{V}_{AB}(\omega)+\tilde{V}_{BC}(\omega)+\tilde{V}_{CA}(\omega)=0$$

Similarly for KCL: $\sum_{node}\tilde{I}_n(\omega)=0$

Note: the amplitudes $\tilde{V}$ and $\tilde{I}$ are complex quantities, but measured V and I are always real →

$\tilde{V}(-\omega)=\tilde{V}^*(\omega),\ \tilde{I}(-\omega)=\tilde{I}^*(\omega)$

$$V(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{V}(\omega)e^{j\omega t}d\omega=2\Re e\sqrt{2\pi}\int_{0}^{\infty}\tilde{V}(\omega)e^{j\omega t}d\omega$$

absorbing the factor of two into the Fourier amplitude we define: $V(t)\equiv\Re e\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\tilde{V}(\omega)e^{j\omega t}d\omega$

Don't panic! You won't need to solve such integrals to understand the circuits you are building here... But is is helpful to have a basic understanding of what it means to look at a signal in the frequency domain.

Example of physical interpretation of the transform

It is easiest to understand the Fourier transform is in the case of periodic functions. Any periodic function can be represented in terms of the Fourier series of sin and cos functions.

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos(n\omega_0 t)+b_n\sin(n\omega_0 t)]$$

where:

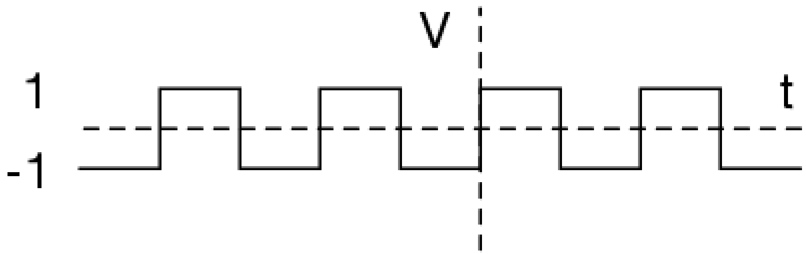
$$a_n=\frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega_0 t)\,dt$$

$$b_n=\frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega_0 t)\,dt$$

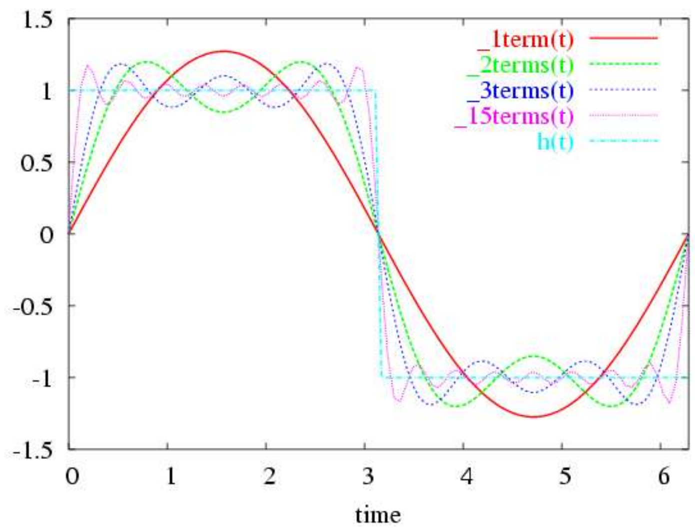
$T\equiv$ period of repitition $\quad \omega_0=2*\pi/T$

The integral form used above applies to non-periodic functions (limit as T approaches infinity).

The figure below show an example of using the series to decompose a periodic square wave with period T=2π. The square wave decomposes into the following Fourier components:



$$f(t)=\frac{4}{\pi}\left[\frac{\sum_{n=1}^{\infty}\sin(nt)}{n}\right]$$



Next we apply the frequency domain formalism to the defining equations for capacitors and inductors.

1) Capacitor: $$\frac{dV}{dt}=\frac{I}{C} \rightarrow \frac{d}{dt}\int \tilde{V}(\omega)e^{j\omega t}d\omega=\int j\omega\tilde{V}(\omega)e^{j\omega t}d\omega=\frac{1}{C}\int \tilde{I}(\omega)e^{j\omega t}d\omega$$
$$\rightarrow \tilde{V}(\omega)=\tilde{I}(\omega)\frac{1}{j\omega C}$$

$\frac{1}{j\omega C}\equiv X_C$ The ***capacitive reactance***

2) Inductor: $V=L\frac{dI}{dt} \rightarrow \tilde{V}(\omega)=\tilde{I}(\omega)j\omega L$

$j\omega L\equiv X_L$ The ***inductive reactance***

In general we use the term ***impedance***, ***Z***, to describe either resistance, reactance, or a combination of both.

Notice that in terms of specific frequencies linearity has been restored to the equations defining our voltage and current relationships. This leads to the ***generalized form of Ohm's Law*** : V = IZ or I = YZ

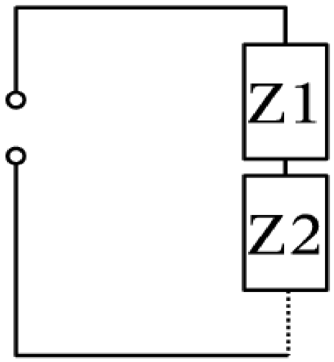
Z ≡ the (generally) frequency dependent impedance (generalization of Resistance)

Y ≡ the (generally) frequency dependent admittance (generalization of Conductance)

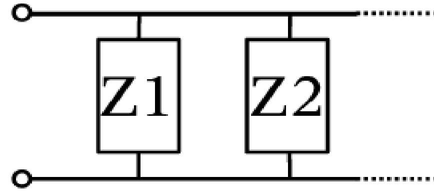
KCL and KVL are also satisfied.

From this point we will omit the "~" notation when referring to the Fourier amplitude. It will be obvious from the context when we are talking about the frequency dependent performance of a circuit.

AC networks may be analyzed using identical techniques to the DC case. But we will implicitly be restricted to a single frequency and the solutions for V's and I's will be the Fourier amplitudes at that frequency.

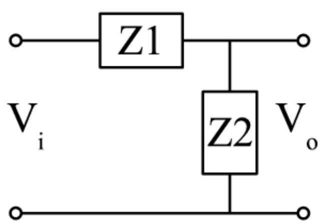


$$Z_{TOTAL}=Z_1+Z_2+\ldots$$



$$\frac{1}{Z_{TOTAL}}=\frac{1}{Z_1}+\frac{1}{Z_2}+\ldots$$

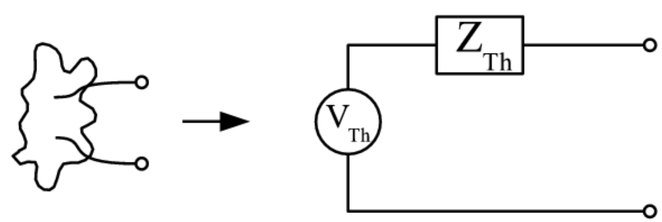
Generalized Voltage Divider



$$V_o=\frac{Z_2}{Z_1+Z_2}V_i$$

Generalized Thevenin Theorem also holds for linear circuits with reactive components

$$V_{Th}(\omega)=V_{O.C.} \ @\ \omega \quad Z_{Th}(\omega)=\frac{V_{O.C.}}{I_{S.C.}} \ @\ \omega$$

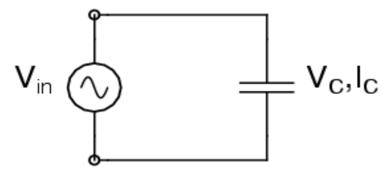


Complex notation in electronics

In a complex, linear circuit driven by a sinusoidal source all currents and voltages in the circuit will be sinusoidal. These currents and voltages will oscillate with the same frequency as the source and their magnitudes will be proportional to the magnitude of the source at all times. The phases of the currents and voltages in the circuit will likely be shifted relative to the source. This is a consequence of the reactive elements in the circuit. When using the generalized forms of Ohm's / Kirchoff's Laws phase changes for AC signals are often important.

The superposition principle allows us to solve for voltage or current for individual driving frequencies, the total voltage or current of interest is simply the superposition of all sinusoidal components of the driving source added back together with the amplitude and phase modifications caused by traversing the circuit.

First consider an AC signal driving a capacitor:



$$V_{\text{in}} = A\cos(\omega t) = V_C$$

$$I_C = C\frac{dV}{dt} \rightarrow I_C = A\omega C\sin(\omega t) = A\omega C\cos(\omega t - 90^\circ)$$

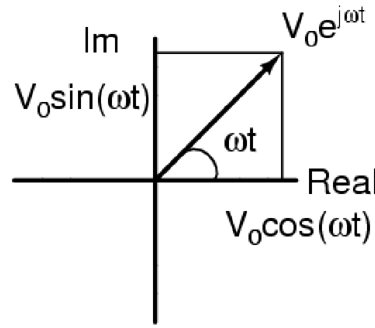
The current is 90 degrees out of phase with voltage. (Lags the voltage by 90 degrees)

Since both voltage magnitudes and phases are generally affected by circuits with complex impedances, it is useful to treat V, I as complex quantities as well in order to keep track of both magnitude and phase. Physical voltages and currents are always real, the complex notation that follows is used as an aid for calculations, but only the real part of a complex voltage or current is used to represent physical quantities.

Start with a sinusoidal voltage: $V = V_0\cos(\omega t)$ convert this into a complex expression as follows: $V = V_0\cos(\omega t) + jV_0\sin(\omega t)$

The imaginary term is has no physical significance, we add this term to allow us to express the voltage in exponential notation (via Euler's relation $V = V_0 e^{j\omega t}$ Physical voltages are recovered by taking the real part of the complex expression. We'll see the advantage of this notation below.

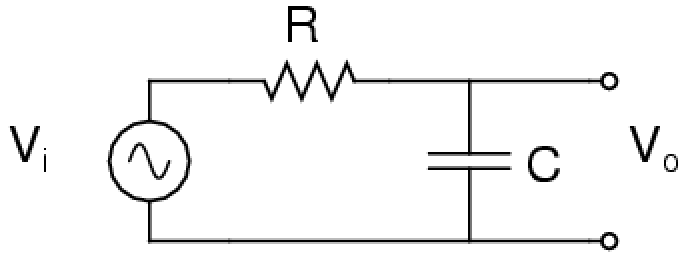
Graphically the voltage can be represented on the complex plane as follows:



The magnitude vector rotates with frequency ω. It is clear that the real component (projection onto the real axis) of the voltage is our input voltage. The phase angle of the voltage, ϕ, at any time is:

$$\phi=\omega t=\tan^{-1}\{\frac{\operatorname{Im} V}{\operatorname{Re} V}\}$$

Let's use this notation to analyze the ***low pass RC filter*** we saw above.



Using the generalized Ohm's law, this circuit is a frequency dependent voltage divider:

$$\frac{V_o}{V_i}=\frac{Z_C}{Z_R+Z_C}=\frac{1/j\omega C}{R+1/j\omega C}=\frac{1}{1+j\omega RC}$$

at DC (ω=0) $V_o = V_i$, for large ω V_o approaches 0

disregarding the phase we can relate the output and input magnitudes: $\left|\frac{V_o}{V_i}\right|=\frac{1}{\sqrt{1+\omega^2R^2C^2}}=A$

for the phase relationship:

$$V_o=\frac{V_i}{1+j\omega RC}=\frac{V_i}{1+\omega^2R^2C^2}(1-j\omega RC)=V_iAe^{-j\phi}$$

where $\phi=\tan^{-1}(-\omega RC)$ (phase angle for the divider circuit)

substitute: $V_i=V_0e^{j\omega t}$

$$V_o=V_0e^{j\omega t}Ae^{-j\phi}=AV_0e^{j(\omega t-\omega RC)}$$

the physical voltage $V_o=\operatorname{Re} V_o=AV_0\cos(\omega t-\omega RC)$

Note: the output voltage has acquired a change in amplitude and a change in phase. Both depend on the frequency.

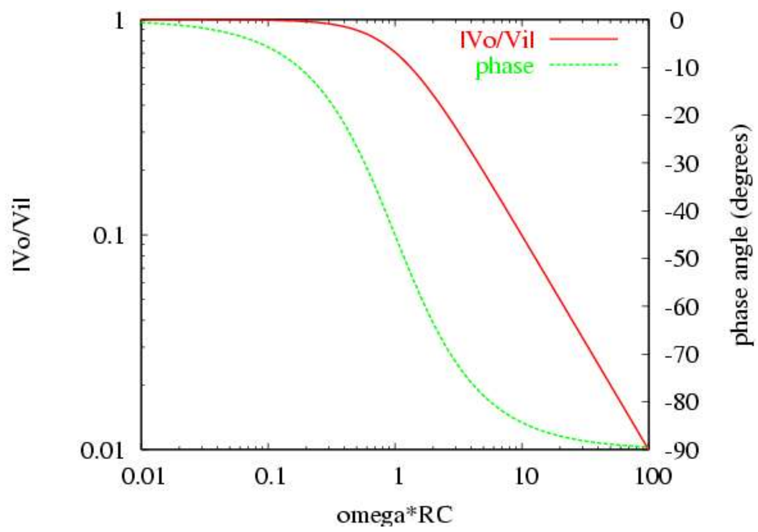
$$\omega\ll\frac{1}{RC}\quad A\to1\quad\text{and}\quad\phi\to0$$

$$\omega\gg\frac{1}{RC}\quad A\to\frac{1}{\omega RC}\quad\text{and}\quad\phi\to-\pi/2$$

Rule of thumb: large phase shifts accompany large attenuations.

Plot of amplitude and phase shift versus frequency:

$$\omega_{3dB} = \frac{1}{RC} \quad f_{3dB} = \frac{\omega_{3dB}}{2\pi}$$



At the breakpoint frequency: $\phi_{3dB} = \frac{-\pi}{4}$

$$\left|\frac{V_o}{V_i}\right| = \frac{1}{\sqrt{2}} = .707$$

The expression for $\frac{V_o}{V_i}$ is called the ***Transfer Function*** of the network.

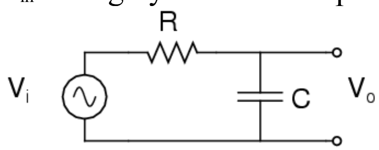
The ***Gain or Attenuation*** $\left|\frac{V_o}{V_i}\right|$ is conventionally expressed in units of decibels (dB). Gain in dB $\equiv 20\log_{10}(\left|\frac{V_o}{V_i}\right|)$

$\left\|\frac{V_o}{V_i}\right\|$	Gain (dB)
0.707	-3dB
0.5	-6dB
0.1	-20dB
0.01	-40db

In the attenuation region (beyond the 3dB point) the output falls at 6dB/octave or 20dB/decade

Input and output impedances

Z_{in} – roughly a circuit's impedance to "ground" seen by a source driving the circuit (impedance to ground looking into circuit's input)



for the low pass filter:

$$Z_{\text{in}} = V/I = R + 1/j\omega C$$
$$\rightarrow R \ \text{ for } \ \omega \gg 1/RC$$
$$\rightarrow \infty \ \text{ for } \ \omega \ll 1/RC$$

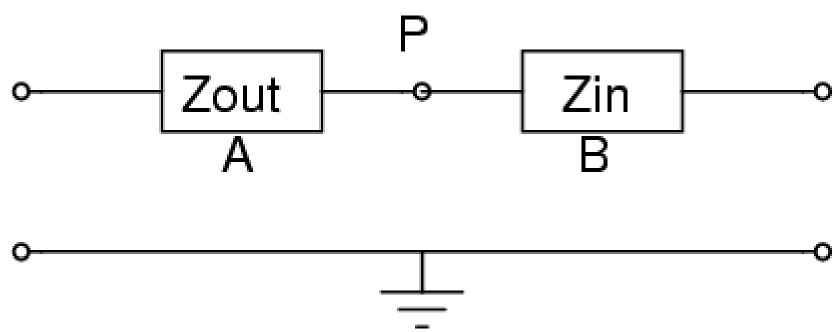
Z_{out} – measure of circuit's ability to drive a load (roughly impedance to "ground" looking into circuit's input). Z_{out} follows from Thevenin's Theorem ($Z_{out} = Z_{th}$):

$$V_{O.C.} = \frac{V_i}{1 + j\omega RC} \qquad Z_{\text{out}} = \frac{V_{O.C.}}{I_{S.C.}} = \frac{R}{1 + j\omega RC}$$
$$I_{S.C.} = \frac{V_i}{R}$$

If Z_{in} of circuit element B is infinite, then $V_P = V_{OC}$

If Z_{in} of circuit element B = Z_{out} of A, then $V_P = V_{OC}/2$

In order for a circuit to drive a load without significant signal attenuation we require: $Z_{in} >> Z_{out}$. In this limit circuit element B will not significantly perturb the performance of element A.



High pass filter / differentiator

$$\frac{V_o}{V_i}=\frac{R}{R+1/j\omega C}=\frac{j\omega RC}{1+j\omega RC}$$

$$\left|\frac{V_o}{V_i}\right|=\frac{\omega RC}{\sqrt{1+\omega^2 R^2 C^2}} \qquad \tan^{-1}(1/\omega RC)$$

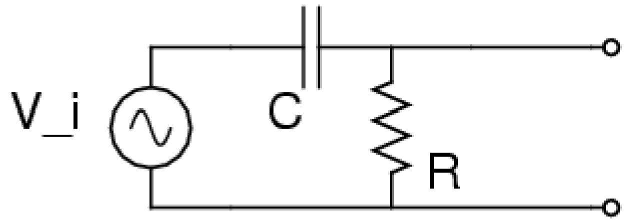
$$\omega \ll 1/RC \qquad \left|\frac{V_o}{V_i}\right|\rightarrow 0 \text{ and } \phi\rightarrow\pi/2$$

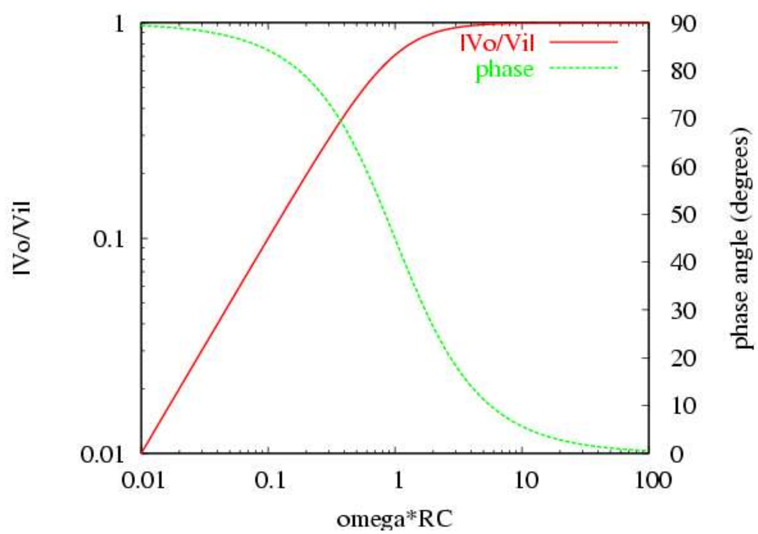
$$\omega \gg 1/RC \qquad \left|\frac{V_o}{V_i}\right|\rightarrow 1 \text{ and } \phi\rightarrow 0$$

$$Z_{\text{in}}=R+1/j\omega C$$

$$Z_{\text{out}}=\frac{R}{1+j\omega RC}$$

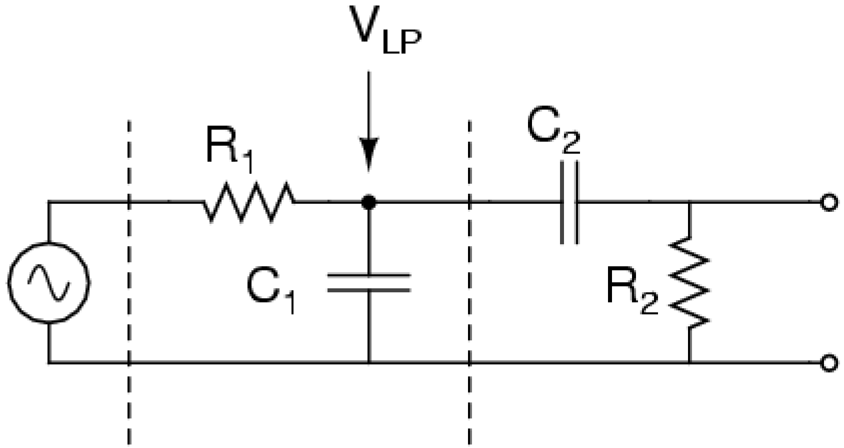
same as low pass filter (but V_Thev is different)

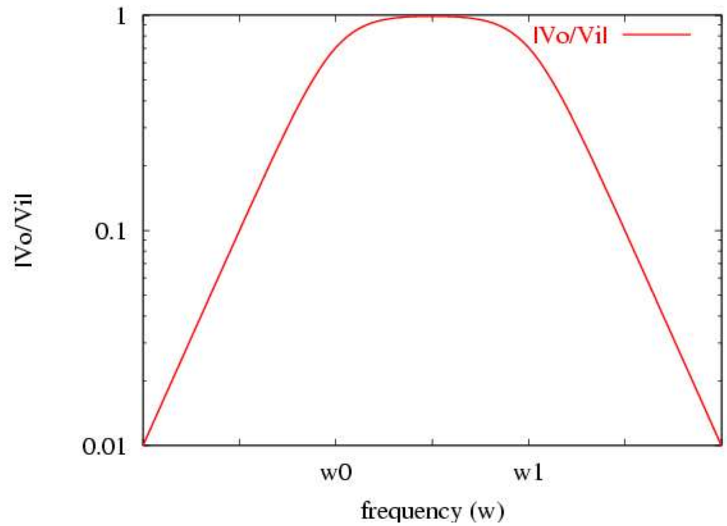




Band pass filters

The importance of calculating input and output impedances of the circuits above is clear when one attempts to combine a High Pass and Low Pass filter to make a bandpass circuit.





Notice this filter is composed of a low pass filter followed by a high pass filter. In general, the response of the low pass filter is modified by the addition of the second filter, because not all of the current flowing through R1 passes through C1. Instead some current is diverted through the high pass filter which acts as a load on the low pass section.

In general we cannot simply multiply the transfer functions for the low and high pass filters to get the combined transfer function. But, in the limit $Z_{\text{in}}^{\text{High Pass}} \gg Z_{\text{out}}^{\text{Low Pass}}$ we approximately recover the simple solution.

For the low pass filter section: $V_{Th} = \frac{V_i}{1 + j\omega R_1 C_1}$ $Z_{Th} = \frac{R_1}{1 + j\omega R_1 C_1} = Z_{out}$ Note: $Z_{out} < R_1$ for all ω.

Choose $min(Z_{\text{in}}^{\text{High Pass}}) = R_2 \gg R_1 \rightarrow V_{LP} \simeq \frac{V_i}{1 + j\omega RC}$ i.e. For all ω this ensures $Z_{\text{in}}^{\text{High Pass}} \gg Z_{\text{out}}^{\text{Low Pass}}$

As a rule of thumb we would choose $R_2 \geq 10 R_1$ (Typically we'll use a factor of 10 to satisfy a ">>" relation.)

The high pass network responds to this input in the usual way.

$V_o = V_{\text{LP}} \frac{j\omega R_2 C_2}{1 + j\,omegaR_2 C_2} \simeq V_i \frac{j\omega R_2 C_2}{(1 + j\omega R_1 C_1)(1 + j\omega R_2 C_2)}$ And the output is just the product of the individual transfer functions.

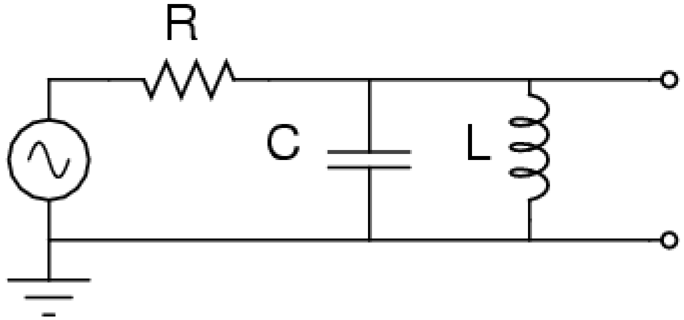
To design a bandpass filter follow these steps:

1) choose $R_1 C_1 = \frac{1}{\omega_1}$

2) let $R_2 \geq 10 R_1$

3) choose C_2 so that $R_2 C_2 = \frac{1}{\omega_2}$

Resonant band pass circuit

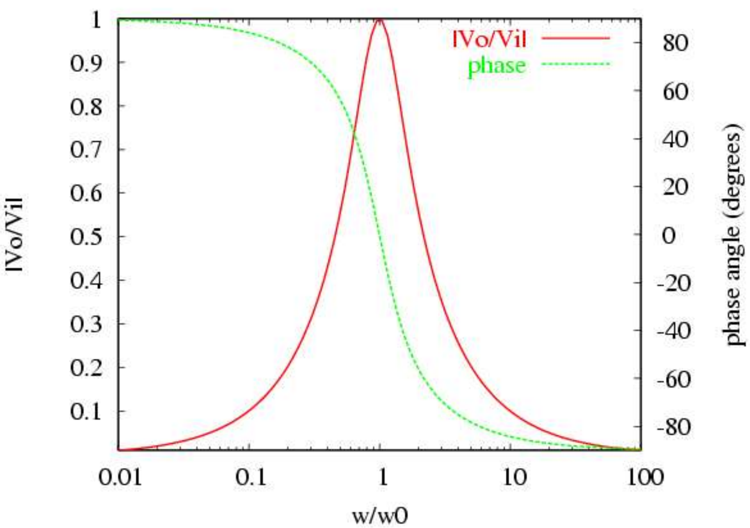


$$Z_{LC}=Z_L \| Z_C=\frac{1}{\frac{1}{j\omega L}+j\omega C}=\frac{L/C}{j\omega L+1/j\omega C}$$

$$Z_{LC}=\frac{R_0}{j(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega})} \quad \text{where} \quad R_0\equiv\sqrt{L/C} \quad \omega_0\equiv\frac{1}{\sqrt{LC}}$$

$$\frac{V_o}{V_i}=\frac{Z_{LC}}{R+Z_{LC}}=\frac{1}{1+j\frac{R}{R_0}(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega})^2}$$

$$|\frac{V_o}{V_i}|=\frac{1}{\sqrt{1+Q^2(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega})^2}} \quad \tan(\phi)=Q(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}) \quad \text{where } Q\equiv\frac{R}{R_0}$$



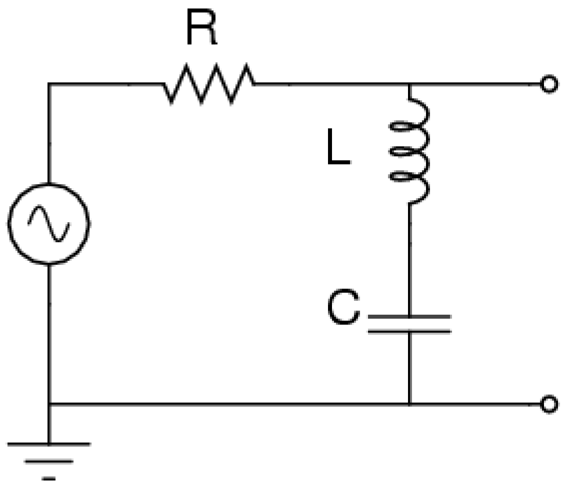
For every $\omega'<\omega_0$ there corresponds an $\omega''>\omega_0$ such that $|V_o(\omega')/V_i(\omega')|=|V_o(\omega'')/V_i(\omega'')|$. By superposition one can show that $\omega'\omega''=\omega_0^2$

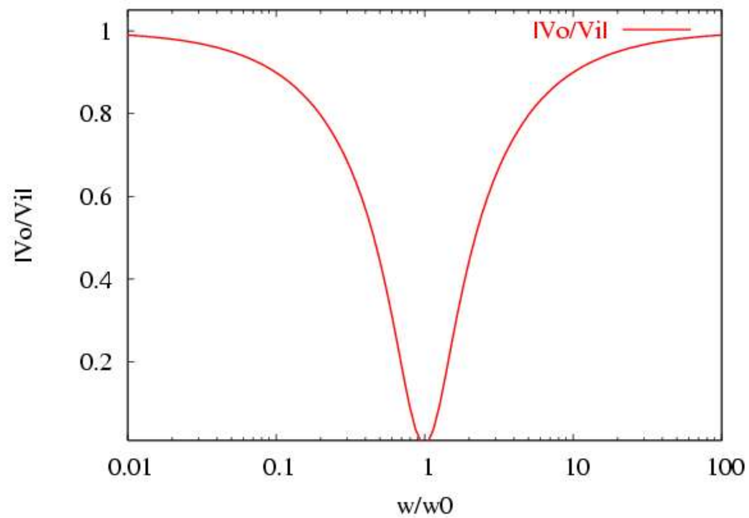
Define two 3dB points: $|\frac{V_o(\omega_1)}{V_i(\omega_1)}|=0.707=|\frac{V_o(\omega_2)}{V_i(\omega_2)}|$

The Bandwidth is defined as $B=\frac{\omega_2-\omega_1}{2\pi}=\frac{1}{2\pi}(\omega_2-\frac{\omega_0^2}{\omega_2}) \quad using\, \omega_1\omega_2=\omega_0^2$

$$at\,\omega_2 \quad Q\left(\frac{\omega_2}{\omega_0}-\frac{\omega_0}{\omega_2}\right)=1=\frac{Q}{\omega_0}\left(\omega_2-\frac{\omega_0^2}{\omega_2}\right) \;\rightarrow\; B=\frac{1}{2\pi}\frac{\omega_0}{Q} \qquad Q=R\sqrt{\frac{C}{L}}=\frac{R}{\omega_0 L}$$

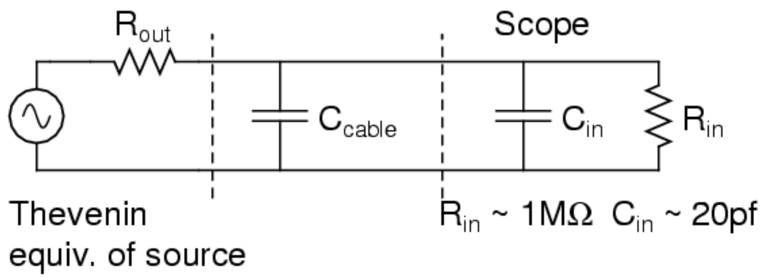
It is also possible to build a ***trap*** or ***notch filter***:





The scope probe (a frequency independent voltage divider)

When a simple piece of coaxial cable is used to connect a circuit to an oscilloscope we are creating a low pass filter in the following manner:



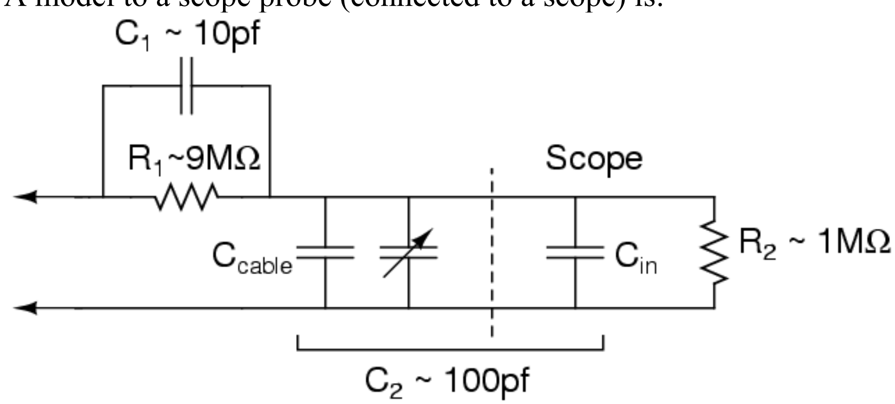
Cable behaves as capacitance of ~ 30pf/foot if not properly terminated. (more on this when we discuss transmission lines)

This circuit forms a low pass filter with: $\tau \simeq R_{out}(C_{\text{cable}}+C_{\text{in}})$ for example if $R_{out}=100\,K, C=100\,pf \rightarrow f_{3\text{dB}}=16\,kHz$

(Perform in-class test using signal generator + 100K resistor)

The scope traces for high output impedance circuits can be very biased if there is no compensation for the low pass filter formed with the cable and scope input.

A model to a scope probe (connected to a scope) is:



Notice: a series capacitor is added at the tip of the probe and a variable capacitor is added at the base of the probe for compensation

Analysis of the probe circuit

$$\frac{V_o}{V_i}=\frac{R_2\|C_2}{(R_1\|C_1)+(R_2\|C_2)}=\frac{\dfrac{R_2}{1+j\omega R_2C_2}}{\dfrac{R_1}{1+j\omega R_1C_1}+\dfrac{R_2}{1+j\omega R_2C_2}}\rightarrow\overbrace{\frac{R_2}{R_1+R_2}}^{\text{if } R_1C_1=R_2C_2}$$

The probe serves as a frequency independent voltage divider once the compensation capacitor is adjusted to satisfy the RC equality above.

The input impedance is: $$Z_{\text{in}}=\frac{R_1}{1+j\omega R_1C_1}+\frac{R_2}{1+j\omega R_2C_2}\rightarrow\begin{array}{ll} R_1+R_2 & \text{as } \omega\rightarrow 0 \\ \dfrac{1}{j\omega}\left(\dfrac{1}{C_1}+\dfrac{1}{C_2}\right) & \text{as } \omega\rightarrow\infty\end{array}$$

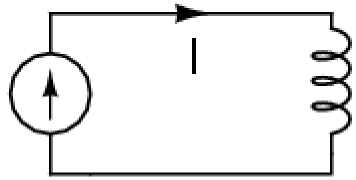
Therefore, compared to the scope + cable, the addition of the scope probe

1) ***increases*** the input resistance $R_2\rightarrow R_1+R_2\simeq 10\,R_2$

2) ***decreases*** the input capacitance $C_2\rightarrow\dfrac{C_1C_2}{C_1+C_2}\simeq\dfrac{1}{10}C_2$

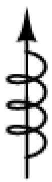
Note: is it important to have $R_1C_1=R_2C_2$, otherwise the response will have a complicated frequency dependence and distorted signals will result.

Transformers



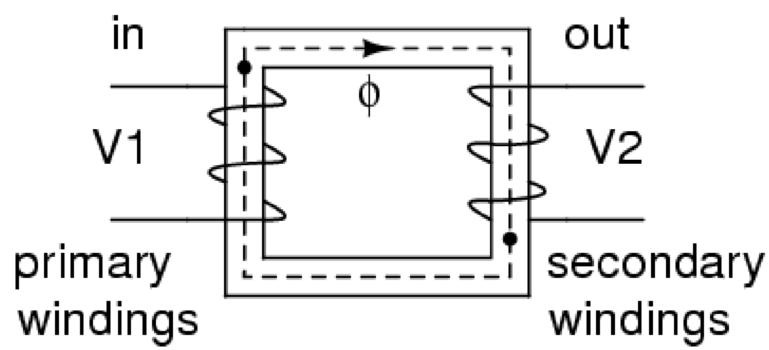
Start with the inductor:

Ohm's Law gives: $V(t)=L\frac{dI(t)}{dt}$



Faraday's Law: change in magnetic flux induces voltage across a coil $V(t)=\frac{-d\phi}{dt}$ where $\phi=B\cdot A$ Field x Area

for multiple turns $V(t)=-N_{turns}\frac{d\phi}{dt}=L\frac{dI(t)}{dt}$



Using magnetic flux to couple two coils:

(“dots” represent in-phase points on the transformer, depends on directions of windings.)

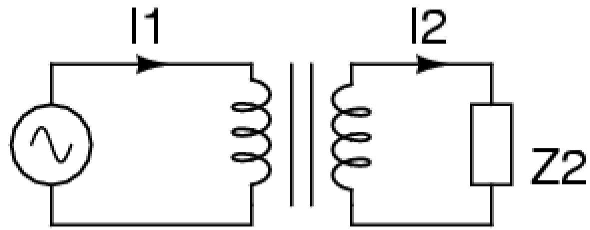
Ideal transformer (100% ϕ coupling)

$V_2=-N_2\frac{d\phi}{dt}$ $V_1=-N_1\frac{d\phi}{dt}$ $\rightarrow\frac{V_2}{N_2}=\frac{V_1}{N_1}$ $\frac{V_2}{V_1}=\frac{N_2}{N_1}$ Ratio of voltages = ratio of turns

Power = V*I is constant across the ideal transformer (Energy in = Energy out), therefore $\frac{I_2}{I_1}=\frac{N_1}{N_2}$

Current goes as inverse of the ratio of turns.

Transforming impedances



impedance Z = V / I

impedance seen by source $Z_1 = \frac{V_1}{I_1}$

load impedance $Z_2 = \frac{V_2}{I_2}$

$$V_2 = V_1 N_2 / N_1 \quad I_2 = I_1 N_1 / N_2$$

substitute $$Z_2 = \frac{V_1}{I_1}\left(\frac{N_2}{N_1}\right)^2 = Z_1\left(\frac{N_2}{N_1}\right)^2 \rightarrow Z_1 = Z_2\left(\frac{N_1}{N_2}\right)^2$$

(Load) impedance seen by source is altered by the transformer. Transformer can allow optimal impedance matching between source and load. Note: maximum power if transferred from source to load in Z_out(source) = Z_in(load).

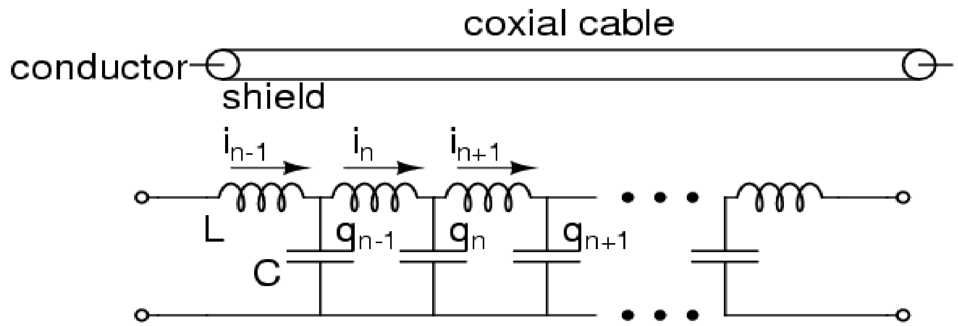
Typical applications for transformers:
1) change V/I levels from on circuit to another
2) impedance matching
3) isolation, i.e. Change of ground reference

Transmission lines

So far we have treated wires and cables as simple, non-interfering elements that "instantly" transmit currents and voltages without significant changes in magnitude. A transmission line is a pair of conductors that carries a signal between two points in a finite time.

A transmission line of length ***l*** has a transmission velocity ***u*** for signals moving in the line. For low frequency signals $\omega \ll u/l$ the voltage is the same on both sides of the cable and we can safely approximate the line as in infinite speed wire. For high frequency $\omega > u/l$ and the voltages will be different on each end of the cable. In this case there will be measurable effects due to the length of the cable.

A two conductor cable can be modeled as in the picture below. All conductors possess a small amount of self inductance per unit length. In the cable there is also stray capacitance between the shield and conductor. In this model we will neglect the small series resistance of the conductor. The conductor can be seen as a large number of small series inductors and a small parallel capacitors. This model is called a ***lumped constant*** LC circuit.



Applying KVL and KCL:

start with $L\frac{di_n}{dt}=\frac{1}{C}q_{n-1}-\frac{1}{C}q_n$ then differentiate both sides: $L\frac{d^2 i_n}{dt^2}=\frac{1}{C}\frac{dq_{n-1}}{dt}-\frac{1}{C}\frac{dq_n}{dt}=\frac{1}{C}(i_{n-1}-i_n)-\frac{1}{C}(i_n-i_{n+1})$

replacing L and C with inductance/unit length $l=L/\Delta x$, capacitance/unit length $c=C/\Delta x$

$$\frac{d^2 i_n}{dt^2}=\frac{1}{lc}\left(\frac{1}{\Delta x}\left(\frac{(i_{n+1}-i_n)}{\Delta x}-\frac{(i_n-i_{n-1})}{\Delta x}\right)\right)$$

in the limit small Δx we can write: $\frac{\partial^2 i}{\partial t^2}=\frac{1}{lc}\frac{\partial^2 i}{\partial x^2}$

this is an example of the wave equation with general solution: $i(x,t)=i_1(x-ut)+i_2(x+ut)$ where ***u*** is the velocity $u=1/\sqrt{lc}$ (~2/3 ***c*** for typical cables)

similarly the voltage solution is: $v(x,t)=v_1(x-ut)+v_2(x+ut)$
in each case the two terms represent signals moving forwards and backwards, respectively.

For a sinusoidal waveform, the forward moving signal can be expressed as: $\begin{aligned}v(x-ut)&=V_0 e^{j(x-ut)}\\ i(x-ut)&=I_0 e^{j(x-ut)}\end{aligned}$

The cable impedance can be found by rewriting $L\frac{di_n}{dt}=\frac{1}{C}q_{n-1}-\frac{1}{C}q_n$ after substituting inductance and capacitance per unit length:

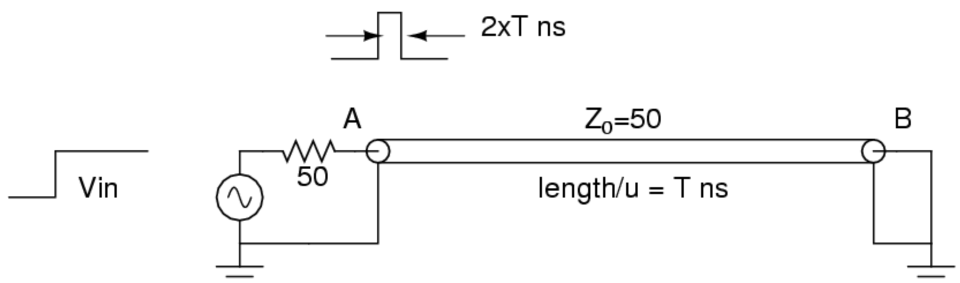
$$\frac{\partial i_n}{\partial t}=-\frac{1}{l}\frac{\partial v}{\partial x}$$

then substituting our solutions for v,i,u yields: $V_0=\sqrt{\frac{l}{c}}I_0$ therefore $Z=\sqrt{\frac{l}{c}}$

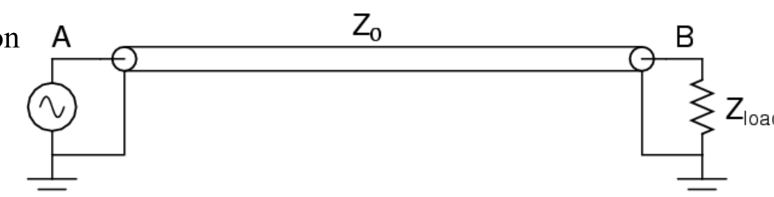
The impedance of the cable only depends on the inductance and capacitance per unit length. It is purely resistive and does not depend on the length of the cable. A transmission can be treated as a device of fixed impedance regardless of length.

Impedances for common cables	
RG-58 coax	50 Ohms
Coax for TV signals	75 Ohms
Flat antenna wire	300 Ohms

Consider a shorted transmission line. A pulse from the voltage source causes a wave to propagates from V_A to V_B in time T. V_B is necessarily fixed to 0V. A wave of opposite phase is created at the short. This appears as a reflection that propagates backward and cancels the signal at V_A.



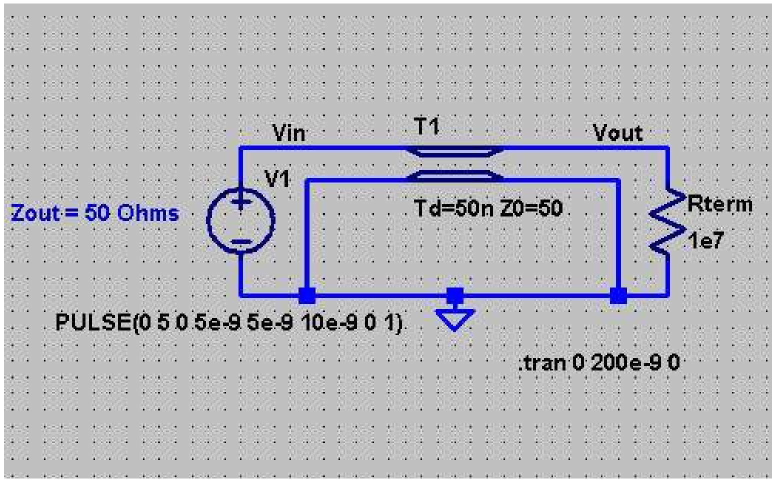
Next consider a line terminated with a resistive load. If $Z_{load} >> Z_0$, the boundary condition at point B is I_B=0. This causes a non-inverted reflection of the signal with amplitude equal to the signal applied at A.



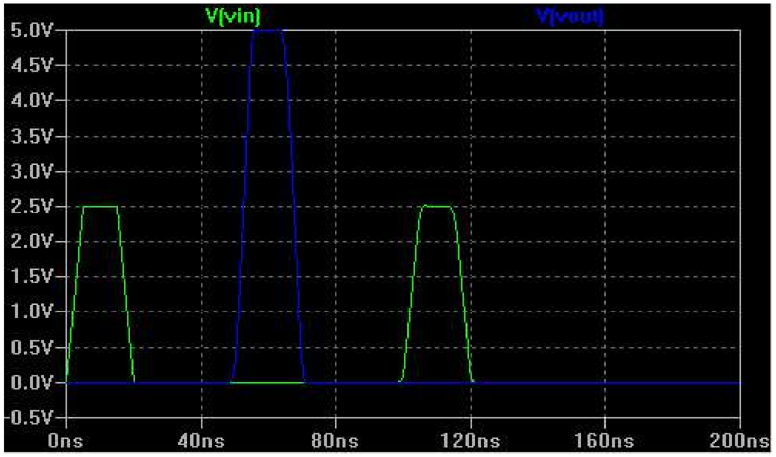
In general the magnitude and sign of the reflected signal is given by:

$$\frac{v_{reflected}}{v_{incident}} = \frac{R_T - Z_0}{R_T + Z_0}$$

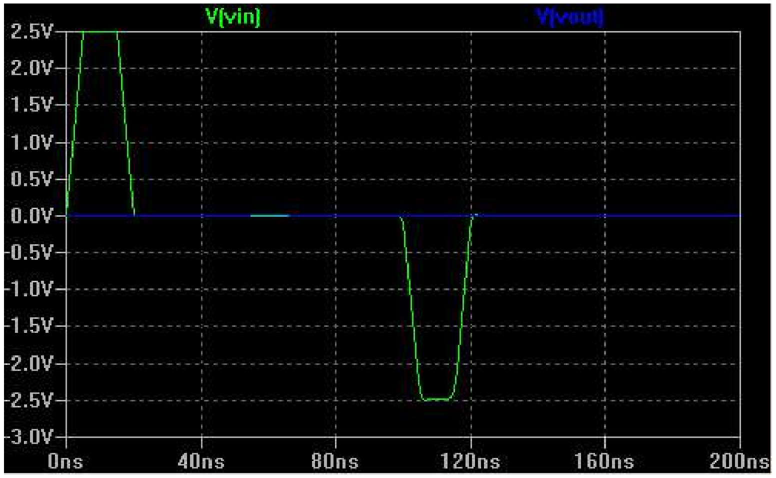
Consider a 50ns long transmission line with a characteristic impedance Z_0 of 50 Ohms. We can define a short pulse as a pulse whose duration is << 50ns and a long pulse as one whose duration is >> 50ns



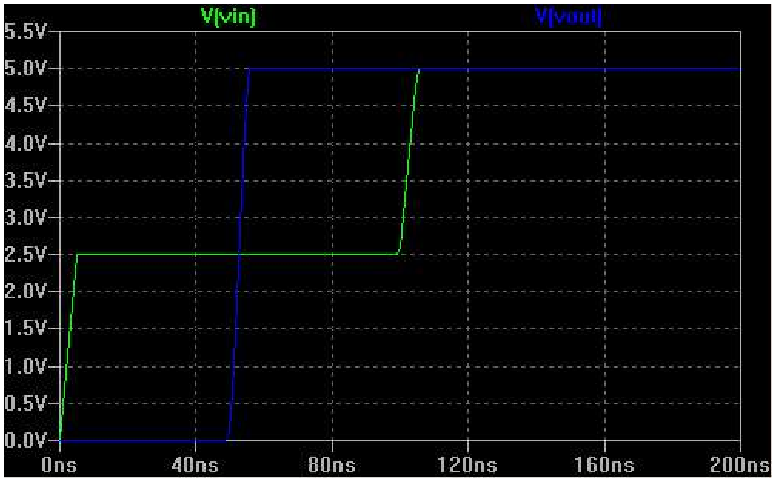
Response to a short pulse for $R_{term} >> Z_0$:



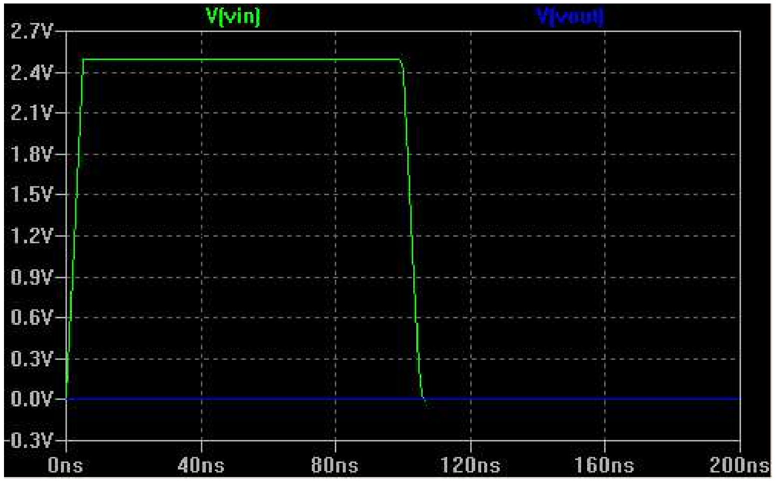
Response to a short pulse for $R_{term} << Z_0$:



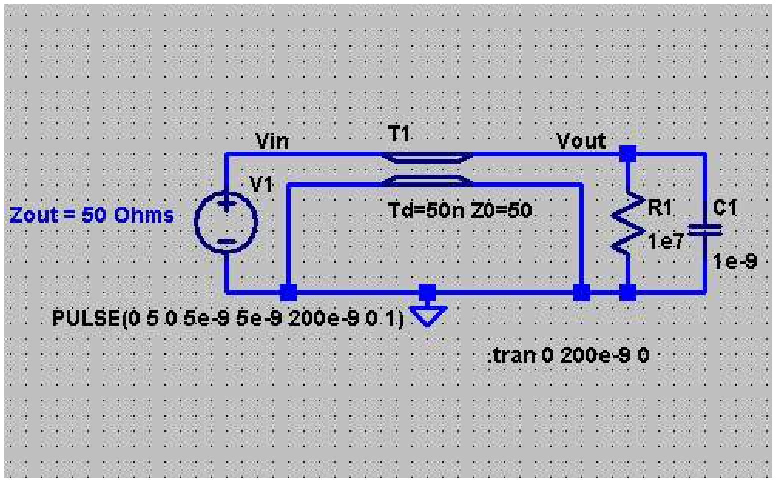
Response to a long pulse for $R_{term} \gg Z_0$



Response to a long pulse for $R_{term} \ll Z_0$



Effect of terminating with a capacitor



$|Z_{load}| = |1/\omega c|$

Charging an unterminated line, RC time constant with steps.