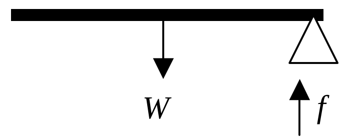
PHYSICS 321 Final Examination (12 December 2002)

Time limit 3 hours. Answer all 6 questions.

1. You and an assistant are holding the (opposite) ends of a long plank when **oops!** the butterfingered assistant drops his end. If the plank's weight is W and if the plank was level when it was dropped, at that (initial) instant the weight you feel is ________ ?



(**Hint:** use Newton's 2nd Law for rotations & linear motion.)

Solution:

$$N = I\ddot{\theta} \qquad \text{(Newton's 2}^{\text{nd}}\text{ Law for rotations)}$$

$$\therefore \quad W\frac{\ell}{2} = \frac{1}{3}M\ell^2\ddot{\theta}$$

$$W - f = Ma = M\frac{\ell}{2}\ddot{\theta} \qquad \text{(Newton's 2}^{\text{nd}}\text{ Law for linear motion)}$$

Eliminating $\ddot{\theta}$ we have

$$W\frac{\cancel{\ell}}{2} = \frac{1}{3}\cancel{M}\cancel{\ell}^{\cancel{2}} \cdot \frac{2}{\cancel{M}\cancel{\ell}}(W - f)$$

$$W = \frac{4}{3}(W - f)$$

$$f = \frac{W}{4}$$

2. The surface gravity of the Moon is almost exactly $1/6$ of the Earth's, and its curvature is such that in 1870 meters of forward progress the surface drops 1 meter. How fast would a projectile launched horizontally have to move in order to make a closed (circular) orbit around the Moon?

(**Hint:** WWGD [what would Galileo do?].)

Solution:

The projectile must fall 1 meter for every 1870 meters of forward motion. Thus,

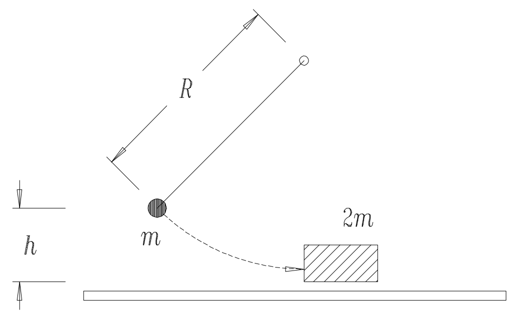
$$\Delta y = \frac{1}{2}g_M(\Delta t)^2 = 1\text{ m}$$

$$\Delta t = \frac{\Delta x}{v} = \frac{1870\text{ m}}{v}$$

and so

$$v = 1870 \cdot \sqrt{\frac{g_M}{2}} = 1870 \cdot \sqrt{\frac{9.8}{12}} = 1.69 \text{ km/sec}.$$

3. A pendulum consists of a massless rigid rod of length R with a weight of mass m attached to its end. The bob is released from a height h above a table, upon which is resting a block of mass $2m$, whose end is exactly below the point of suspension. There is no friction between block and table. The bob collides elastically with the block. To what height does it rise, and does it rebound or continue in the same direction?



Describe the result if the bob had mass $2m$ and the block mass m.

Solution:
Use conservation of kinetic energy and linear momentum to describe the collision. Then

$$\frac{1}{2}\cancel{m}v^2 = \frac{1}{2}\cancel{m}v'^2 + \frac{1}{2}(2\cancel{m})u'^2$$

$$\cancel{m}v = \cancel{m}v' + 2\cancel{m}u'$$

and eliminating u' we find

$$v^2 - v'^2 \equiv (v-v')(v+v') = 2\cdot\left[\frac{1}{2}(v-v')\right]^2 = \frac{1}{2}(v-v')^2.$$

This gives either the uninteresting solution $v = v'$ (no collision) or the interesting one

$$v' = -\frac{v}{3}.$$

The latter represents a rebound ($v' < 0$). Since the velocity of the bob before the collision is given by energy conservation as

$$\frac{1}{2}mv^2 = mgh,$$

we see $h' = \frac{v'^2}{2g} = \frac{h}{9}$. The bob rebounds to 1/9 of its original height.

If the bob weighed $2m$ and the block weighed m, the combination of energy and momentum conservation would give

$$2(v-v')(v+v') = [2(v-v')]^2 = 4(v-v')^2$$

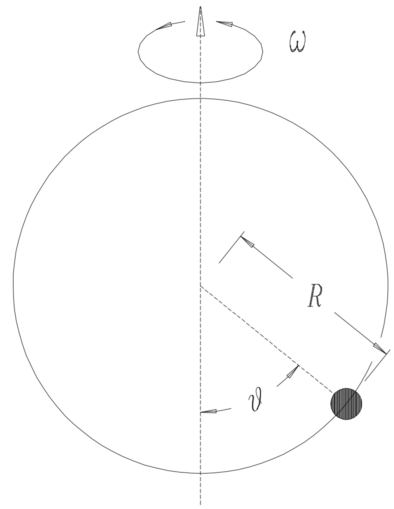
$$v' = \frac{v}{3}, \quad u' = \frac{4}{3}v$$

That is, the bob continues in the same direction at 1/3 of its original speed. (The block moves fast enough to get out of the way!) The bob again rises to height $h' = h/9$.

4. A bead of mass m slides frictionlessly on a hoop of radius R that is oriented vertically and rotates about a diameter with constant angular velocity ω.
   a) Write the Lagrangian of the system in appropriate generalized coordinates, and find the equation(s) of motion.

   b) Show that there is a critical value ω_c of the rotational velocity such that for $\omega < \omega_c$ the equilibrium position is at the bottom center, whereas for $\omega > \omega_c$ the equilibrium position is at a non-zero angle θ.

   c) Examine small oscillations about equilibrium in both cases and determine whether the equilibria are stable or unstable, and if stable, determine their oscillation frequencies.



Solution:

The Lagrangian is

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$
$$= \frac{1}{2}mR^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right) + mgR\cos\theta$$
$$= \frac{1}{2}mR^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) + mgR\cos\theta$$

and the equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = mR^2\ddot{\theta} = -\frac{\partial}{\partial\theta}V_{eff}(\theta) = mR^2\left(-\Omega^2 + \omega^2\cos\theta\right)\sin\theta$$

where $\Omega^2 = g/R$ and

$$V_{eff}(\theta) = -mR^2\left(\Omega^2\cos\theta + \frac{1}{2}\omega^2\sin^2\theta\right).$$

At equilibrium, $\ddot{\theta} = 0$. Thus if $\Omega^2 > \omega^2$ the only equilibria are $\theta = 0, \pi$, whereas if $\Omega^2 < \omega^2$, there is also a third equilibrium point at $\theta = \theta_c = \cos^{-1}\left(\frac{\Omega^2}{\omega^2}\right)$. We conclude that $\omega_c = \Omega = \sqrt{\frac{g}{R}}$.

If we expand $V_{eff}(\theta)$ about the equilibrium points we find

$$V_{eff}(\theta) \approx mR^2\begin{cases} -\Omega^2 + \frac{1}{2}\left(\Omega^2 - \omega^2\right)\theta^2, & \theta \approx 0 \\ \Omega^2 - \frac{1}{2}\left(\Omega^2 + \omega^2\right)(\theta - \pi)^2, & \theta \approx \pi \end{cases}$$

and for $\omega > \omega_c$,

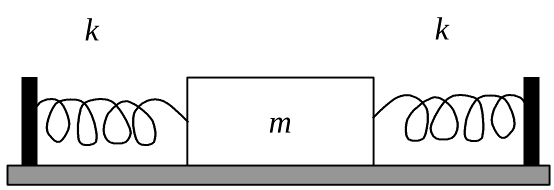
$$V_{eff}(\theta) \approx mR^2\left(-\frac{\Omega^4}{\omega^2} - \frac{1}{2}\left(\omega^2 - \Omega^2\right) + \frac{1}{2}\left(\omega^2 - \Omega^2\right)\left(\theta - \theta_c\right)^2\right).$$

For any ω the equilibrium at $\theta = \pi$ is unstable—the potential has a maximum there.

For $\omega < \omega_c$ the equilibrium at $\theta = 0$ is stable, and the angular frequency of small oscillations about that point is $\sqrt{\Omega^2 - \omega^2}$.

For $\omega > \omega_c$ the equilibrium at $\theta = 0$ is unstable (the coefficient of θ^2 has changed sign, turning a minimum into a maximum), but the equilibrium at $\theta = \theta_c$ is stable. The angular frequency of small oscillations about that point is $\sqrt{\omega^2 - \Omega^2}$.

5. A block of mass *m* is attached to two fixed posts by springs of constant *k* and negligible mass, as shown, and executes simple harmonic motion in the horizontal direction. (It slides frictionlessly on a table.)



a) What is the frequency of oscillation?
b) A lump of putty, also of mass *m*, falls on the block and sticks to it. What is the effect on the amplitude and frequency of the subsequent oscillation if
   1) the lump hits just when the block is at an extreme end of its oscillation;
   2) the lump hits when the block is exactly in the middle of its oscillation?

Solution:
The equation of motion of the oscillator before the putty falls is

$$m\ddot{x} + 2kx = 0$$

so the angular frequency of oscillation is $\omega = \sqrt{\frac{2k}{m}}$.

If the putty falls at the limit of the oscillation, i.e. $x = x_{\max}$, the velocity of the block is 0, so all that happens is the mass, and hence the frequency changes. The new frequency is $\omega' = \sqrt{\frac{2k}{2m}} = \frac{\omega}{\sqrt{2}}$, and the amplitude remains the same.

If the putty falls when $x = 0$ the velocity is maximum. Then, by conservation of linear momentum the new velocity is $v' = \frac{1}{2}v$, so that the kinetic energy falls by a factor of 2 (2× the mass, ¼ of the velocity squared). But at the extrema of the motion, all the energy is potential (KE=0) so that the square of the new amplitude must be half the square of the old. That is, we have $x'_{\max} = \frac{1}{\sqrt{2}} x_{\max}$. The frequency will be the same as in case 1) above.

6. A particle of mass m moving in one dimension is subject to the force

$$F = -F_0 \sinh(ax).$$

a) What is the potential energy corresponding to this force?
b) Does this potential have a stable equilibrium position?
c) If the answer to b) above is "Yes", what is the **period** of small oscillations about equilibrium?
d) If the answer to b) above is "Yes", find an expression for the period of oscillations that are not small. (Do not attempt to evaluate this expression!)

Solution:

Since $F = -\frac{dV}{dx}$, we may immediately integrate and, disregarding a constant of integration, find

$$V(x) = \frac{F_0}{a} \cosh(ax).$$

Since this potential has a single minimum and becomes infinite at $x = \pm\infty$, we see that there is a stable equilibrium position at $x = 0$. If we expand about this point we find

$$V(x) = \frac{F_0}{a} \cosh(ax) \approx \frac{F_0}{a} + \frac{1}{2} \frac{F_0}{a} (ax)^2 = \frac{F_0}{a} + \frac{1}{2} F_0 a\, x^2.$$

The frequency of small oscillations is therefore $\omega = \sqrt{\frac{F_0 a}{m}}$ and the period is

$$\tau = 2\pi \sqrt{\frac{m}{F_0 a}}.$$

To find the period of non-small oscillations, we use conservation of energy:

$$\frac{1}{2} m\dot{x}^2 + \frac{F_0}{a} \cosh(ax) = E = \frac{F_0}{a} \cosh(a\ell)$$

where ℓ is the amplitude of the oscillation. This equation is separable, giving

$$\sqrt{\frac{ma}{2F_0}} \frac{dx}{dt} = \pm \sqrt{\cosh(a\ell) - \cosh(ax)}$$

or

$$\tau = 2\int_0^{\tau/2} dt = 2\sqrt{\frac{ma}{2F_0}} \int_{-\ell}^{\ell} \frac{dx}{\sqrt{\cosh(a\ell) - \cosh(ax)}} = \sqrt{\frac{2m}{aF_0}} \int_{-\ell a}^{\ell a} \frac{du}{\sqrt{\cosh(a\ell) - \cosh(u)}}.$$

The latter integral cannot be evaluated in closed form.