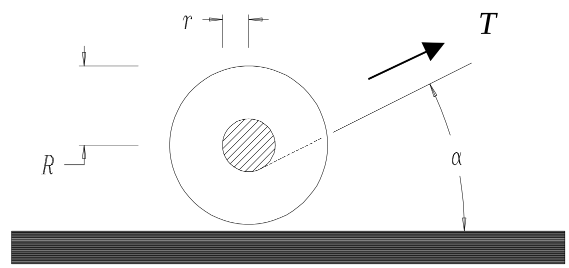
Do all 4 problems. Time: 1 hour.

1. A spool of thread rests on a horizontal table, as shown. The end of the thread is pulled upward at an angle α to the horizontal. The thread passes under the core of the spool, and is pulled with tension T. The core radius is r and the outer radius is R. The spool's mass is M. Assuming the spool rolls without slipping, under what conditions will it roll to the left? (Do not forget that the spool must not be lifted off the table!) Draw and label all forces and torques.



Solution:
There is a leftward-pointing tangential force $-F\hat{x}$ at the point of contact between spool and table. The tension can be resolved as $\vec{T} = \hat{x}T\cos\alpha + \hat{z}T\sin\alpha$. The force of gravity is $-mg\hat{z}$, but the table exerts an upward normal force $N\hat{z}$. Thus Newton's 2nd Law reads

$$m\ddot{x} = -F + T\cos\alpha$$

$$m\ddot{z} = -mg + T\sin\alpha + N$$

and the law for rotational motion about the axis of the spool reads

$$I\dot{\omega} = -FR + Tr\,.$$

Finally, as long as the spool rolls without slipping, $\dot{x} = -R\omega$. (The – sign comes from the convention that counter-clockwise rotations are positive.)

We want $N - m\ddot{z} \geq 0$ so that the spool does not accelerate upward (it remains in contact with the table), hence

$$mg \geq T\sin\alpha\,.$$

We can use Newton's 2nd Law in the *x*-direction to eliminate the unknown force *F*.
Finally, we use $\ddot{x} = -R\dot{\omega}$ to get an equation for $\dot{\omega}$:

$$\left(I + mR^2\right)\dot{\omega} = T\left(r - R\cos\alpha\right).$$

Therefore for the spool to accelerate to the left ($\dot{\omega} > 0$) we must have $\cos\alpha < r/R < 1$.

2. A spacecraft in a circular orbit of radius R about a massive primary applies its engines in the direction of its motion. If the acceleration is so brief as to be considered an instantaneous impulse, and if the speed is increased by a factor α, $v \to \alpha v$, under what conditions on α will the new orbit be:
   a) Elliptical?
   b) Parabolic?
   c) Hyperbolic?

   (**Hint**: use your physical intuition. Do not try to set up and solve the orbit equations!)

Solution:
The orbit is circular, hence centripetal and centrifugal accelerations balance:

$$\frac{MG}{R^2} = \frac{v^2}{R}$$

hence the total energy is

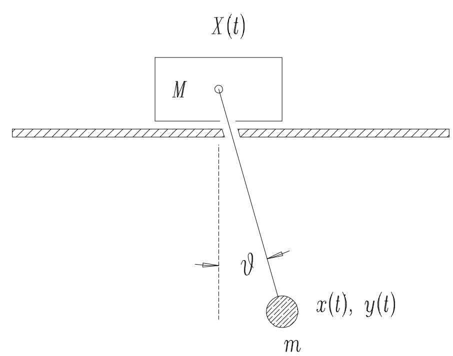
$$-\frac{mMG}{R} + \frac{1}{2}mv^2 \equiv -\frac{mMG}{2R} = E < 0\,.$$

Hence if we increase v by a factor α, the energy becomes

$$E' = -\frac{mMG}{R} + \frac{1}{2}mv^2\alpha^2 = \frac{mMG}{2R}\left(\frac{\alpha^2}{2} - 1\right)$$

For the new orbit to be an ellipse, we must have $E' < 0$, or $\alpha < \sqrt{2}$. To be parabolic requires $E' = 0$ or $\alpha = \sqrt{2}$. And a hyperbola requires $E' > 0$ or $\alpha > \sqrt{2}$.

3. The system shown to the right consists of a block of mass M that can slide without friction, in the x-direction along a horizontal air track. A pendulum hangs from a pivot attached to the block, in such a way that it swings in the x-y plane (y = vertical). The pendulum consists of a massless rod of length r and a bob of mass m.



   a) Write down the Lagrangian of the system in terms of the Cartesian coordinates $X(t)$, $x(t)$ and $y(t)$ of block and pendulum, respectively.

   Solution:
   Let us measure the position of the bob relative to the block. Then the total kinetic energy is

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left(\dot{x} + \dot{X}\right)^2 + \frac{1}{2}m\dot{y}^2$$

   The potential energy is

$$V = mgy\,,$$

and

$$L = T - V = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left(\dot{x}+\dot{X}\right)^2 + \frac{1}{2}m\dot{y}^2 - mgy\,.$$

b) Express the coordinates of part a) in terms of the generalized coordinates $X(t)$ and $\vartheta(t)$ (where the angle ϑ is as shown in the Figure), and re-express the Lagrangian in terms of these coordinates.

Solution:

Clearly,

$$x = r\sin\vartheta$$

$$y = -r\cos\vartheta$$

so

$$L = \frac{1}{2}\left(M+m\right)\dot{X}^2 + mr\cos\vartheta\dot{X}\dot{\vartheta} + \frac{1}{2}mr^2\dot{\vartheta}^2 + mgr\cos\vartheta$$

c) Write the Euler-Lagrange equations of motion for the system. ***Do not waste time solving them.***

Solution:

Since $\dfrac{\partial L}{\partial X} = 0$ the X coordinate is *cyclic*, meaning that

$$\frac{\partial L}{\partial \dot{X}} = \left(M+m\right)\dot{X} + mr\cos\vartheta\dot{\vartheta} \equiv \frac{d}{dt}\left[\left(M+m\right)X + mx\right] = \text{constant}\,.$$

The other equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\vartheta}}\right) - \frac{\partial L}{\partial\vartheta} = 0 = \frac{d}{dt}\left(mr\cos\vartheta\dot{X}\right) + mr^2\ddot{\vartheta} + mgr\sin\vartheta - mr\sin\vartheta\dot{\vartheta}\dot{X}\,.$$

4. Match the conservation laws in the left half of the table below with the symmetries that give rise to them:

Conservation Law	Answer	Symmetry
a) Linear momentum	**2**	1. Time translation invariance
b) Angular momentum	**3**	2. Space translation invariance
c) Energy	**1**	3. Rotation invariance
		4. Galilean invariance
		5. Lorentz invariance
		6. Gauge invariance
		7. Reflection invariance (parity)

That is, in each shaded answer box place the appropriate number from the right-hand list.