PHYS 321 Homework Assignment #8
Solutions:

1. Calculate the principal moments of inertia for the following objects of uniform density:

a) Cylinder.

Solution:
The M-of-I about the symmetry axis is the same as for a disk, namely \( \frac{1}{2}MR^2 \). We can integrate \( y^2 + z^2 \) over the volume to get \( I_{xx} \). \( I_{yy} \) is the same as \( I_{xx} \) by symmetry. The result is \( M \left( \frac{1}{4}R^2 + \frac{1}{12}l^2 \right) \). However, we have seen from the perpendicular axis theorem that the moment of inertia of a disk about an axis lying in the disk and passing through its center is \( \frac{1}{2}MR^2 \). So divide the cylinder into thin disks of mass \( dm = Mdz/\ell \) and note that the moment of inertia of each disk, about the \( x \)-axis passing through the CM of the cylinder, and \( \perp \) the symmetry axis (that is, the \( z \)-axis), is (by the parallel axis theorem)

\[
\int_{-l/2}^{l/2} dI_{xx} = \int_{-l/2}^{l/2} \frac{1}{4}R^2 dm + \int_{-l/2}^{l/2} z^2 dm = \frac{1}{4}MR^2 + \frac{1}{12}M\ell^2 .
\]

b) Cone.

Solution:
To get the M-of-I about the symmetry axis, we divide the cone into disks of radius \( r(z) \) and height \( dz \) and integrate:

\[
I_{zz} = \int_0^h dm \left( \frac{1}{2} r^2 (z) \right)
\]

where

\[
r (z) = \frac{z}{h} R
\]

and

\[
dm = \rho \pi r^2 (z) dz .
\]

The mass is then

\[
M = \int_0^h dm = \frac{1}{3} \rho \pi R^2 h
\]
and
\[ I_{zz} = \frac{1}{2} \rho \pi \int_0^h dz \, r^4 (z) = \frac{3}{10} MR^2. \]

To get \( I_{xx} \equiv I_{yy} \) we use the parallel axis theorem: the M-of-I about an axis passing through the apex and \( \perp \) the symmetry axis is
\[ I_{xx} \text{(apex)} = \int_0^h dm \left( \frac{1}{3} r^2 (z) + z^2 \right) = \frac{3}{20} MR^2 + \frac{3}{5} Mh^2. \]

We may then use the fact the center of mass is located \( \frac{3}{4} \) the distance from the apex to the base. That is, we can apply the parallel axis theorem to get
\[ I_{xx} \text{(apex)} = I_{xx} + M \left( \frac{3}{4} h \right)^2 = I_{xx} + \frac{9}{16} M h^2 \]
or
\[ I_{xx} = \frac{3}{20} MR^2 + \frac{3}{80} M h^2. \]

c) Long thin rod.
Solution:
This one we did in class. We have \( I_{xx} = I_{yy} = \frac{1}{12} M \ell^2 \) and \( I_{zz} = 0 \).

d) Sphere.
Solution:
For a sphere we see that by symmetry \( I_{xx} = I_{yy} = I_{zz} \). We need to integrate
\[ I_{zz} = \rho \int d^3 r \left( x^2 + y^2 \right) \]
\[ = \frac{3M}{4\pi R^3} \int_0^{2\pi} d\varphi \int_0^R dr \, r^2 \int_0^\pi d\theta \, \sin \theta \, r^2 \sin^2 \theta \]
\[ = \frac{3M}{2R^3} \left( \frac{R^5}{5} \right) \left( \frac{4}{3} \right) = \frac{2}{5} MR^2. \]

e) Spherical shell of inner radius \( r \) and outer radius \( R \).
Solution:
For this one we may add the moment of inertia of a sphere of
mass $M + m$ and radius $R$ to that of a concentric sphere of mass $-m$ and radius $r$ to get

$$I = \frac{2}{5} \left[ (M + m) R^2 - mr^2 \right]$$

$$= \frac{2}{5} \left( \frac{4\pi\rho}{3} \right) \left[ (R^3) R^2 - (r^3) r^2 \right] = \frac{2}{5} M \left( \frac{R^5 - r^5}{R^3 - r^3} \right).$$

We see that for a thin shell we get the answer in the book.

2. Evaluate the inertia tensor of a cube, using a reasonable set of principal axes. What is the moment of inertia of the cube about an axis connecting a corner to its diagonal opposite (and passing through the center of the cube)? In fact, what is the $M$ of $I$ about any axis passing through the center of the cube?

**Solution:**

We take the principal axes to be normal to the faces and passing through the geometric center of the cube. Now, the moment of inertia of a thin square slab about an axis parallel to the sides of the square and passing through its center is

$$dI_z = \rho \ell dy \int_{-\ell/2}^{\ell/2} dx x^2 = \frac{1}{12} \rho \ell^4 dy.$$

Then using the $||$ axis theorem we have

$$I_z = \int_{-\ell/2}^{\ell/2} (dI_z + \rho \ell^2 y^2 dy) = \frac{1}{12} \rho \ell^5 + \frac{1}{12} \rho \ell^5 = \frac{1}{6} M \ell^2.$$

By symmetry, all the moments are identical. Hence the moment of inertia of a cube about any axis through the CM is the same,

$$I = \frac{1}{6} M \ell^2.$$

That is,

$$I_{\mu\nu} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \equiv I \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
so that a rotation gives

\[ P'_{\mu\nu} = \mathcal{O}^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathcal{O} \]

\[ = I \mathcal{O}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathcal{O} \]

\[ = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \]

This is of course trivially true for a sphere or in fact any solid whose principal moments are equal.

3. A thin uniform rigid rod of mass \( m \) and length \( \ell \), initially not rotating and aligned along the \( y \)-direction and moving at velocity \( \vec{v} = v\hat{x} \) in a plane as shown below, strikes a small immovable peg, as shown. (The collision takes place at the bottom end of the rod.) If the collision is elastic (energy is conserved), describe the motion of the rod after the collision. What if the collision is completely inelastic (bottom end recoils at velocity \( \vec{v}' = 0 \))? Describe the motion and calculate the energy loss in the latter case.

\[ \vec{v} \]

\[ \bullet \]

**Solution:**

The elastic collision conserves energy and angular momentum. Equating initial and final angular momenta gives

\[ -\frac{mv\ell}{2} = -\frac{mv'\ell}{2} - \frac{1}{12}m\ell^2 \omega'. \]

Doing the same for initial and final kinetic energies we have

\[ \frac{1}{2}mv^2 = \frac{1}{2}mv'^2 + \frac{1}{2}I\omega'^2 = \frac{1}{2}mv'^2 + \frac{1}{24}m\ell^2 \omega'^2. \]
We find, therefore, that

\[ u' \overset{df}{=} \ell \omega' = \begin{cases} 
0 \\
3v 
\end{cases} \]

and

\[ v' = \begin{cases} 
v \\
\frac{1}{2}v 
\end{cases} \]

That is, there are two solutions, one where the rod misses and continues as before; the other in which the cm continues to move at half the original speed in the same direction, and the rod acquires a clockwise rotation at angular velocity \( \omega' = -3v/\ell \). Note that the instantaneous velocity of the lower end of the rod, as seen in the frame of the peg, immediately following the collision is \(-v\), exactly as we would expect for an elastic collision between a mass and an immovable body. Note that 25% of the original kinetic energy is in the linear motion and 75% in the rotational motion.

The problem could also have been solved by imagining the peg exerted an impulse \(-\Delta p\) on the rod. Then conservation of momentum gives

\[ mv - \Delta p = mv' \]

and conservation of angular momentum gives

\[ -\frac{\ell}{2} \Delta p = -\frac{1}{12} m \ell^2 \omega' = -\frac{1}{12} m \ell u'. \]

Imposing the condition that the lower end of the rod recoils at \(-v\) (elasticity condition) we have

\[ -\frac{\ell}{2} \omega' + v' \equiv -\frac{1}{2} u' + v' = -v \]

so that

\[ v - v' = \frac{1}{m} \Delta p = \frac{1}{6} u' \]

and

\[ v + v' = \frac{1}{2} u' \]

leading to the same (non-trivial) solution.
To do the second part of the problem (inelastic collision), we impose the condition that the instantaneous velocity of the bottom end of the rod immediately post-collision is 0, so that combining momentum and angular momentum conservation we have

\[ v - v' = \frac{1}{m} \Delta p = \frac{1}{6} u' \]

but now the condition relating \( u', v' \) and \( v \) is

\[ -\frac{1}{2} u' + v' = 0 \]

so that

\[ E' = \frac{1}{2} m v'^2 + \frac{1}{24} m \ell^2 \omega'^2 \]

\[ = \frac{1}{2} m v^2 \left[ \left( \frac{3}{4} \right)^2 + \frac{1}{12} \left( \frac{3}{2} \right)^2 \right] = \frac{3}{4} E. \]

The rod continues to move at cm velocity three-fourths of its original velocity, but spins with half the angular velocity of the elastic collision, again in the clockwise direction. Now 2/3 of the (residual) kinetic energy is in the forward motion and only 1/3 in the rotational motion.

4. A physical pendulum consists of a rod of mass \( m \) and length \( \ell \) to the end of which is attached a solid sphere of mass \( M \) and radius \( r \), constrained to swing from a pivot, in a vertical plane. (That is, the distance from the pivot to the top of the spherical bob is \( \ell \).) What is the period of small oscillations?

**Solution:**

We use the parallel axis theorem to compute the M-of-I of the pendulum. We have

\[ I = I_{rod} + I_{sphere} + M (\ell + r)^2 = \frac{1}{3} m \ell^2 + \frac{2}{5} M r^2 + M (\ell + r)^2 \]

and the center of mass of the pendulum is located a distance

\[ \lambda = \frac{m}{2 (M + m)} \ell + \frac{M}{M + m} (\ell + r) \]

from the pivot. The equation of motion is thus

\[ I \ddot{\theta} + (M + m) g \lambda \sin \theta = 0 \]
5. Another physical pendulum consists of a cone suspended by its apex and confined to swing in the $x-z$ plane ($z$ vertical). What is its period for small displacements from equilibrium?

\[
\tau = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{(M + m) g \lambda}} = 2\pi \sqrt{\frac{\frac{1}{2}m\ell^2 + \frac{2}{5}Mr^2 + M (\ell + r)^2}{g[m\ell + M (\ell + r)]}}.
\]

Solution:
The CM of the cone is $3/4$ the distance from apex to base, hence for a cone of height $h$ we have an equation of motion

\[I_{xx}\ddot{\theta} + \frac{3}{4}Mgh\sin\theta = 0\]

(where $I_{xx}$ is the moment of inertia about an axis through the apex and perpendicular to the symmetry axis) giving the period of small oscillations

\[
\tau = 2\pi \sqrt{\frac{4I_{xx}}{3Mgh}} = 2\pi \sqrt{\frac{4 \left(\frac{3}{8}MR^2 + \frac{3}{5}Mh^2\right)}{3Mgh}} = 2\pi \sqrt{\frac{R^2 + 4h^2}{5gh}}.
\]