PHYS 321 Homework Assignment #9
Due: Friday, 22 November 2002 (4 probs)

   Solution:

   The situation is as shown above. We take the body-fixed axes $\hat{x}'$ and $\hat{y}'$ to lie in the plane of the sheet, and let the (constant) angular velocity $\vec{\omega}$ define the space-fixed axis $\hat{z}$. Then since the angular velocity lies in the $\hat{x}'-\hat{y}'$ plane—that is, along the line joining the diagonals—we may write

   $$\vec{\omega} = \omega \left[ \frac{2}{\sqrt{3}} \hat{x}' + \frac{1}{\sqrt{3}} \hat{y}' \right].$$

   We note that in the body frame,

   $$\vec{L} = \vec{I} \cdot \vec{\omega} = \frac{2\omega}{\sqrt{3}} I_{xx} \hat{x}' + \frac{\omega}{\sqrt{3}} I_{yy} \hat{y'}$$

   is also constant. The Euler equations then become

   $$\vec{\dot{L}} = \frac{d\vec{L}}{dt}_{\text{body}} + \vec{\omega} \times \vec{L} \equiv \vec{\omega} \times \vec{L} = \frac{2\omega^2}{3} (I_{yy} - I_{xx}) \hat{z},$$

   where we define the body $\hat{z}'$ axis to the perpendicular to the sheet. Now the difference of the two principal moments of inertia is

   $$I_{yy} - I_{xx} = \frac{ma^2}{3} - \frac{ma^2}{12} = \frac{ma^2}{4}$$
so that the (time-dependent) torque, as measured in space-fixed coordinates, is
\[ \vec{N} = \frac{ma^2 \omega^2}{10} (\hat{x} \cos \omega t + \hat{y} \sin \omega t) \]
\[ = a \frac{\sqrt{5}}{2} \vec{z} \times ( \vec{F}_2 + \vec{F}_1 ) = a \sqrt{5} \vec{z} \times \vec{F}_2 \]
where we use the fact that by symmetry the forces at the bearings must be equal in magnitude and opposite in direction (also the net force must add to zero because there is no acceleration of the center of mass). The space-fixed axes \( \hat{x} \) and \( \hat{y} \) are clearly perpendicular to the shaft, hence to \( \vec{\omega} \). We then write
\[ \vec{F}_2 = \alpha \hat{x} + \beta \hat{y} \]
and see immediately that
\[ \vec{F}_2 = \frac{ma \omega^2}{10 \sqrt{5}} (\hat{x} \sin \omega t - \hat{y} \cos \omega t) = -\vec{F}_1. \]

2. Problem 7.17. (Note: the dimensions of the raquet are given in Eq. 7.114.)

**Solution:**
Neglecting the inertia of the strings, we see that the M of I of the raquet about the 1 axis is
\[ I_1 = \frac{1}{2} m_a a^2 \]
since the contribution of the handle about the same axis is 0. Similarly, the M of I of the raquet about the 2 axis is
\[ I_2 = \frac{1}{2} m_a a^2 + m_a R^2 + \frac{1}{12} m_d \ell^2 + m_d \left( a + \frac{\ell}{2} - R \right)^2 \]
Finally (by the perpendicular axis theorem) the third principal moment is \( I_3 = I_1 + I_2 \), hence it is always the largest.

As we saw in class, the distance \( s \) of the center of percussion from the CM is
\[ s = \frac{I_2}{M x} \]
where $x$ is the distance from the point of suspension (grip) to the CM. If the distance $s$ is to be $R = 17.5\text{ cm}$ then we must have $x \approx 20\text{ cm}$, taking the value of $I_2$ in Eq. 7.115 (which, by the way, is slightly wrong; using the numbers in 7.114 I get $1.18 \times 10^{-2}$). Since the distance from the end of the handle to the CM is $\ell + a - R = 33.5\text{ cm}$, the player must be holding the racket about 13 cm from the end, which is a vry choked-up sort of grip. So in fact, as far as I can tell, the design of this racket is wrong, contrary to the statement of B&O.

3. The Earth’s axis is tilted, relative to its orbital plane, by an angle of about 23 deg. The Earth can be regarded as an oblate (squashed) sphere, with an equatorial bulge described by the mass-density

$$\rho(r, \theta) = \rho_0 \Theta(R(\theta) - r)$$

where

$$\Theta(x) = \begin{cases} 
0, & x < 0 \\
1, & x > 0 
\end{cases}$$

and

$$R(\theta) = R_E[1 - \varepsilon P_2(\cos \theta)]$$

Use the fact that the difference of the moments of inertia $I_z - I_x$ is about 1/300 of their average (b&O p. 259) to estimate the parameter $\varepsilon$. Then calculate the torque on the Earth due to the Sun’s gravitational attraction. Finally, estimate the rate of precession of the North Pole. (Precession of the equinoxes.)

**Solution:**

![Equation Diagram]

Where:

- $r$: Distance from the point of suspension (grip) to the CM.
- $s$: Distance from the end of the handle to the CM.
- $R$: 17.5 cm.
- $\ell$: The distance from the point of suspension to the CM.
- $a$: The distance from the end of the handle to the CM.
- $R_E$: Earth's equatorial radius.
- $\varepsilon$: The parameter to estimate.
- $P_2$: Legendre polynomial of degree 2.

Equation:

$$\rho(r, \theta) = \rho_0 \Theta(R(\theta) - r)$$

Where $\Theta(x)$ is a step function.

$$R(\theta) = R_E[1 - \varepsilon P_2(\cos \theta)]$$
We need to calculate several things about the planet. First let us calculate its mass to \( \mathcal{O}(\varepsilon) \) we have

\[
m = \int d^3 r \rho(r, \theta) = \rho_0 \int d\hat{r} \int_0^{R(r, \theta)} r^2 dr = \frac{1}{3} \rho_0 R_0^3 \int d\hat{r} (1 - \varepsilon P_2 (\hat{r} \cdot \hat{z}))^3 \\
\approx \frac{1}{3} \rho_0 R_0^3 \int d\hat{r} (1 - 3\varepsilon P_2 (\hat{r} \cdot \hat{z})) = \frac{4\pi}{3} \rho_0 R_0^3 + \mathcal{O}(\varepsilon^2).
\]

Next we calculate its principal moments. Taking the z-axis to be the planet’s spin axis, we have

\[
I_z = \int d^3 r \rho(r, \theta) (x^2 + y^2) = \int d^3 r \rho(r, \theta) r^2 \sin^2 \theta \\
= \frac{1}{5} \rho_0 R_0^5 \int d\hat{r} (1 - \varepsilon P_2 (\hat{r} \cdot \hat{z}))^5 \left(1 - (\hat{r} \cdot \hat{z})^2\right) \\
\approx \frac{1}{5} \rho_0 R_0^5 \int d\hat{r} (1 - 5\varepsilon P_2 (\hat{r} \cdot \hat{z})) \left(\frac{2}{3} - \frac{2}{3} P_2 (\hat{r} \cdot \hat{z})\right) \\
= \frac{8\pi}{15} \rho_0 R_0^5 (1 + \varepsilon) = \frac{2}{5} m R_0^2 (1 + \varepsilon).
\]

Similarly,

\[
I_x = \frac{1}{10} \rho_0 R_0^5 \int d\hat{r} (1 - \varepsilon P_2 (\hat{r} \cdot \hat{z}))^5 \left(1 + (\hat{r} \cdot \hat{z})^2\right) \\
\approx \frac{1}{10} \rho_0 R_0^5 \int d\hat{r} (1 - 5\varepsilon P_2 (\hat{r} \cdot \hat{z})) \left(\frac{4}{3} + \frac{2}{3} P_2 (\hat{r} \cdot \hat{z})\right) \\
= \frac{8\pi}{15} \rho_0 R_0^5 \left(1 - \frac{\varepsilon}{2}\right) = \frac{2}{5} m R_0^2 \left(1 - \frac{\varepsilon}{2}\right).
\]

Since \( 3\varepsilon/2 \approx 1/300 \), we have \( \varepsilon \approx 1/450 \).

Now we must calculate the potential as a function of the position in the orbit and the orientation of the planet’s axis as it revolves around its primary. The potential is

\[
V = -M_\odot G \int d^3 r \frac{\rho(r, \hat{r} \cdot \hat{z})}{|\hat{R} - \hat{r}|}
\]

which, because \( R \gg r \), can be expanded as

\[
V = -\frac{M_\odot G}{R} \int d^3 r \rho(r, \hat{r} \cdot \hat{z}) \sum_{\lambda=0}^{\infty} \left(\frac{r}{R}\right)^\lambda \lambda P_\lambda (\hat{r} \cdot \hat{R}).
\]
As usual we perform the radial integral first, to get

\[ V = -\frac{M_\odot G}{R} \rho_0 a_0^3 \sum_{\lambda=0}^{\infty} \left( \frac{r_0}{R} \right)^\lambda \frac{1}{\lambda + 3} \int d{\hat{r}} P_\lambda \left( {\hat{r}} \cdot \hat{R} \right) \left[ 1 - \varepsilon P_2 \left( {\hat{r}} \cdot \hat{z} \right) \right]^\lambda + 3 \]

and then note that

\[ \int d{\hat{r}} P_\lambda \left( {\hat{r}} \cdot \hat{R} \right) P_{\lambda'} \left( {\hat{r}} \cdot \hat{z} \right) = \frac{4\pi}{2\lambda + 1} P_\lambda \left( \hat{z} \cdot \hat{R} \right) \delta_{\lambda\lambda'} \]

so that to \( O(\varepsilon) \) we have

\[ V = -\frac{mM_\odot G}{R} + \frac{3}{2} \varepsilon I M_\odot G \frac{P_2 \left( \hat{z} \cdot \hat{R} \right)}{R^3}, \]

where, since the principal moments differ only by \( O(\varepsilon) \), \( I \) is the moment of inertia of the planet, taken as a sphere.

Next, let us suppose the orbit is circular, and let

\[ \hat{R} = \hat{\xi} \cos \Omega t + \hat{\eta} \sin \Omega t; \]

then choose \( \hat{\xi} \) (unit vector in space-fixed \( x \)-direction) such that

\[ \hat{z} \cdot \hat{R} = \sin \theta \cos \Omega t. \]

Here \( \theta \) is the inclination of the Earth’s axis from the normal to the orbital plane. Defining

\[ \Gamma = \frac{3}{2} \varepsilon I M_\odot G \]

the torque is then

\[ \hat{N} = -\frac{\partial V}{\partial \theta} = -3\Gamma \cos^2 \omega t \sin \theta \cos \phi \hat{\phi}; \]

that is, the couple (forces \( \pm \hat{F} \) shown in the Figure above) acts to reduce the inclination angle, and produces a torque perpendicular to the body \( \hat{z} \) axis. We could apply the Euler equations,

\[ \left. \frac{d\hat{L}}{dt} \right|_{\text{body}} + \hat{\Omega} \times \hat{L} = \hat{N}, \]

to determine the precession; alternatively we may use the Lagrangian method as with the heavy top (done in class and in B&O), taking as
the effective Lagrangian in body coordinates (we ignore the free-fall toward the primary)

\[ \mathcal{L} = \frac{1}{2} I_x \left( \dot{\theta}^2 + \dot{\psi}^2 \sin^2 \theta \right) + \frac{1}{2} I_z \left( \dot{\psi} + \dot{\phi} \cos \theta \right)^2 - \Gamma P_2 \left( \sin \theta \cos \Omega t \right). \]

The resulting equation for the Euler angle \( \theta \) is

\[ I_x \ddot{\theta} = I_x \dot{\psi}^2 \sin \theta \cos \theta - I_x \omega \sin \theta \dot{\phi} - 3 \Gamma \sin \theta \cos \theta \cos^2 \Omega \]

and we find (assuming, as in class, that \( \theta \) is constant in time) the “small” precession rate

\[ \dot{\phi} \approx \left( \frac{3}{2} \right)^2 \epsilon \cos \theta \frac{M_\odot G}{R^3 \omega} \equiv \left( \frac{3}{2} \right)^2 \epsilon \cos \theta \frac{\Omega^2}{\omega}, \]

where we have used the fact that since the Earth’s orbital frequency \( \Omega \) is vastly greater than the precession rate, we may replace \( \cos^2 \Omega t \) by its time average, 1/2. We also have used Kepler’s Third Law to eliminate \( M_\odot G \) and the Earth-Sun distance \( R \).

If we evaluate this we find \( \dot{\phi}_0 \approx 8.61 \times 10^{-5} \text{ rad/yr} \), giving a period for the equinoctial precession of about 80,000 years. A similar formula applies for the precession due to the Moon’s gravitational influence; here we have

\[ \dot{\phi}_{\text{Moon}} \approx \left( \frac{3}{2} \right)^2 \epsilon \cos \theta \frac{M_{\text{Moon}} G}{R^3 \omega} \equiv \left( \frac{3}{2} \right)^2 \epsilon \cos \theta \frac{\Omega^2_{\text{Moon}}}{81 \omega} \]

\[ \approx 1.66 \times 10^{-4} \text{ rad/yr}. \]

Combining the two yields an equinoctial precession period of about 25,000 years. The influence of the major planets, especially Jupiter and Saturn, increases the precession still further. The observed period is about 21,000 years. In the 4,000 years or so that have elapsed since the signs of the Zodiac were allotted their monthly periods of influence, the equinoxes have precessed about 1/6 of the way around the Earth’s orbit, meaning the signs are off by at least 2 months.

4. A uniform wire is stretched between two fixed supports 1 meter apart. If the tension in the wire is 500 Nt and the lineal density is 0.0065 gm/cm, what is the frequency of the lowest resonance?
Solution:
The equation of motion of the string is

$$\mu \frac{\partial^2 \psi}{\partial t^2} - T \frac{\partial^2 \psi}{\partial x^2} = 0$$

where $\mu$ is the mass per unit length and $T$ the tension. The perpendicular displacement of the string is $\psi(x, t)$. It is easy to see that the time-dependence is sinusoidal, i.e. the substitution

$$\psi(x, t) = e^{i\omega t} \varphi(x)$$

turns the equation into a first-order ordinary differential equation,

$$\varphi''(x) + \frac{\mu}{T} \omega^2 \varphi(x) = 0$$

where $''$ means the second derivative with respect to $x$. We recognize this as the harmonic oscillator equation, whose general solution is $A \sin(kx) + B \cos(kx)$. The displacement must vanish at the pegs holding the string, i.e. at $x = 0$ and $x = L$. Thus we see $B = 0$ (from the first condition) and then since $\sin(kL) = 0$, we must have $kL = n\pi$ where $n$ is a positive integer. However, by substitution we see that

$$k = \omega \sqrt{\frac{\mu}{T}}$$

so that the resonant angular frequencies are

$$\omega_n = \frac{n\pi}{L} \sqrt{\frac{T}{\mu}} = 2\pi \nu_n$$

or

$$\nu_n = \frac{n}{2L} \sqrt{\frac{T}{\mu}}.$$

Substituting the parameters given, we have $\nu_1 = 438.5$ Hz.