PHYS 321 Lecture 16
Hanging Chain – Lagrange Multipliers  
10-4-02

General:
Take Function $f(q_1, \ldots, q_n)$ with constraint $g(q_1, \ldots, q_n) = 0$
$q_n = h(q_1, \ldots, q_{n-1})$

Differentiating:
$$\frac{\partial f}{\partial q_k} + \frac{\partial f}{\partial q_n} \frac{\partial h}{\partial q_k} = 0 \quad \text{and } k = 1, \ldots, n-1$$

$$\frac{\partial g}{\partial q_k} + \frac{\partial g}{\partial q_n} \frac{\partial h}{\partial q_k} = 0 \quad \text{because } g(\ldots) = 0$$

$$\frac{\partial h}{\partial q_k} = -\frac{\partial g}{\partial q_k} \left( \frac{1}{\frac{\partial g}{\partial q_n}} \right)$$
and then plug back in to get:

$$\frac{\partial f}{\partial q_k} - \frac{\partial f}{\partial q_n} \left( \frac{\partial g}{\partial q_n} \right)^{-1} \frac{\partial g}{\partial q_k} = 0 \quad \text{where } \lambda \text{ is the Lagrange Multiplier}$$

now:
$$\left[ \frac{\partial f}{\partial q_k} - \lambda \frac{\partial g}{\partial q_k} \right] + \left[ \frac{\partial f}{\partial q_n} - \lambda \frac{\partial g}{\partial q_n} \right] \frac{\partial h}{\partial q_k} = 0 \quad \frac{\partial h}{\partial q_k} \text{ can not always be zero}$$

$$\frac{\partial f}{\partial q_k} - \lambda \frac{\partial g}{\partial q_k} = 0 \quad \text{and } \frac{\partial f}{\partial q_n} - \lambda \frac{\partial g}{\partial q_n} = 0$$

only need $\frac{\partial f}{\partial q_k} - \lambda \frac{\partial g}{\partial q_k} = 0$ and $g(q_1, \ldots, q_n) = 0$

if more than one constraint
$g_s(q_1, \ldots, q_n), s = 1, \ldots, m$
then
$$\frac{\partial f}{\partial q_k} - \sum_{s=1}^{m} \lambda_s \frac{\partial g}{\partial q_k} = 0$$
Hanging Chain Problem:

Trying to minimize the potential energy $V$

$$ds = \sqrt{(dx)^2 + (dz)^2} \quad \text{and}$$

$$dm = \mu ds \quad \text{where } \mu \text{ is the mass per unit length}$$

$$dV = dmgz$$

\[ V = \mu g \int_0^L z(x) \sqrt{1 + \left(\frac{dz}{dx}\right)^2} \, dx \quad \text{and} \quad l = \int_0^L ds(x) = \int_0^L dx \sqrt{1 + z'^2} \]

$\delta (V - \mu g \lambda L) = 0$  use of $\mu g$ in the Lagrange multiplier allows it to be factored out in the next step

$$\mu g \int_0^L (z - \lambda) \sqrt{1 + z'^2} \, dx$$

substitution $u = z - \lambda$

,u' = z'$

$$\delta \int_0^L u \sqrt{1 + u'^2} \, dx$$

$$\sqrt{1 + u'^2} - \frac{\partial}{\partial x} \left( \frac{uu'}{\sqrt{1 + u'^2}} \right) = 0$$

integrating factor

$$uu' - \frac{uu'}{\sqrt{1 + u'^2}} - \frac{\partial}{\partial x} \left( \frac{uu'}{\sqrt{1 + u'^2}} \right) = 0$$

$$\frac{\partial}{\partial x} \left( \frac{\sqrt{1 + u'^2}}{u} \right) - \frac{\partial}{\partial x} \left( \frac{uu'}{\sqrt{1 + u'^2}} \right) = 0$$

$$u'^2 \left( 1 - \frac{u'^2}{1 + u'^2} \right) = \alpha^2 \quad \text{because this must be a positive number}$$

$$1 \quad = \frac{\alpha^2}{1 + u'^2}$$

$$u^2 = \frac{u^2}{\alpha^2} - 1$$

$$u = \alpha \cosh(\theta) \rightarrow \quad u' = \alpha \sinh(\theta) \quad \theta' = \frac{u'^2}{\alpha^2} \Rightarrow \quad u^2 = \alpha^2 \sinh^2(\theta) \Rightarrow \quad \sinh^2(\theta) - 1$$
\[ \theta' = \frac{1}{\alpha} \rightarrow \theta = \frac{x}{\alpha} + \beta \]

Therefore the equation of the chain's height is

\[ z(x) = \lambda + \alpha \cosh\left(\frac{x}{\alpha} + \beta\right) \]

to find the minimum you have to take the first derivative

\[ z' = \sinh\left(\frac{x}{\alpha} + \beta\right) = 0 \]

Now the boundary conditions are

\[ \begin{align*}
x = 0, & \quad z = h_1 \rightarrow z(0) = h_1 = \lambda + \alpha \cosh(\beta) \\
x = L, & \quad z = h_2 \rightarrow z(L) = h_2 = \lambda + \alpha \cosh\left(\frac{L}{\alpha} + \beta\right) \
\end{align*} \]

\[ l = \text{constant} \quad \rightarrow \quad l = \int_0^L dx \sqrt{1 + \sinh\left(\frac{x}{\alpha} + \beta\right)} \]

\[ = \int_0^L dx \cosh\left(\frac{x}{\alpha} + \beta\right) \]

\[ = \alpha \left[ \sinh\left(\frac{L}{\alpha} + \beta\right) - \sinh(\beta) \right] \]

Now have 3 equations with 3 variables so the problem could be taken to a numerical answer

hint:

If \( h_1 = h_2 \) the problem simplifies some as \( \beta = -\frac{L}{2\alpha} \)