Lecture 2 (30 August 2002)

Kinematics vs. dynamics

Kinematics is the mathematical description of motion. It was first invented by Galileo (1564-1642). René Descartes (1596-1650) seems either to have invented the idea independently or gotten it from Galileo’s description of the motion of a projectile, in Dialogue on Two New Sciences.

We imagine we have a 3-dimensional Cartesian coordinate system, as shown to the right, although in fact the coordinate system need not be Cartesian. The position of a particle is described by a vector \( \mathbf{r}(t) \) which can be variously written as

\[ \mathbf{r}(t) = x(t) \hat{x} + y(t) \hat{y} + z(t) \hat{z} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} . \]

Galileo also first discussed motion at constant velocity and at constant acceleration. Today we understand the operational definition of velocity as

\[ \mathbf{v}_{av} = \frac{\mathbf{r}(t+\delta t) - \mathbf{r}(t)}{\delta t} \]

for some \( \delta t > 0 \). The mathematical definition of velocity is the time derivative of position, imagining the latter to be a vector of differentiable functions:

\[ \mathbf{v} = \frac{d\mathbf{r}(t)}{dt} = \lim_{\delta t \to 0} \frac{\mathbf{r}(t+\delta t) - \mathbf{r}(t)}{\delta t} . \]

Similarly acceleration is defined operationally as the change of velocity in a finite time, divided by that time; or mathematically as the time-derivative of velocity, assuming the latter to be a vector of differentiable functions.

Mostly we do not distinguish between our operational definitions (appropriate to measurements in a laboratory) and the mathematical definitions, but it is as well to remember that they are different.

Example:

We suppose we know the acceleration of a particle at all times, and that we know its initial position and initial velocity. Then we may write

\[ \mathbf{v}(t) = \mathbf{v}(0) + \int_0^t dt' \mathbf{a}(t') , \]

\footnote{…that is, the three directions are at right angles to each other, and do not change orientation from point to point in space (as, e.g., the three directions do in spherical coordinates).}
and
\[ \mathbf{r}(t) = \mathbf{r}(0) + \int_0^t \mathbf{v}(t') \, dt'. \]
Suppose
\[ \mathbf{a} = \text{constant} \]
so we can perform the integrals easily: we get
\[ \mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}t \]
\[ \mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{a}t^2. \]
For the specific case of a cannonball fired at angle \( \theta \) to the horizontal, we have
\[
\begin{pmatrix}
  x(t) \\
  y(t) \\
  z(t)
\end{pmatrix} = \begin{pmatrix}
  x_0 + (v_0 \cos \theta) t \\
  y_0 \\
  z_0 + (v_0 \sin \theta) t - \frac{1}{2} gt^2
\end{pmatrix}
\]
where we take the motion entirely in the \( x \) (horizontal) and \( z \) (vertical) directions.

If we transform to the coordinate \( x(t) \) we find
\[ z(x) = z_0 + \tan \theta (x - x_0) - \frac{g}{2(v_0 \cos \theta)^2} (x - x_0)^2 \]
which is the equation of a parabola, as Galileo noted. It is easy to see that if \( \tan \theta \geq 0 \) (the missile is projected at or above the horizontal) the range is (we want \( z_f = 0 \))
\[ x_{\text{max}} = x_0 + \left( \frac{v_0 \cos \theta}{g} \right)^2 \left( \tan \theta + \sqrt{\tan^2 \theta + \frac{2z_0 g}{(v_0 \cos \theta)^2}} \right). \]
If the missile is projected from zero initial height, we have
\[ R = \frac{v_0^2}{g} 2 \sin \theta \cos \theta = \frac{v_0^2}{g} \sin (2\theta) \]
whose maximum occurs at 45° elevation, as everyone knows.