Lecture 5: 06 September 2002

Work and kinetic energy

Begin with Newton’s Second Law

\[ \mathbf{F} = m \frac{d\mathbf{v}}{dt}. \]

Work is defined by

\[ dW = \mathbf{F} \cdot d\mathbf{r}, \]

hence

\[ dW = m \frac{d\mathbf{v}}{dt} \cdot d\mathbf{r}, \]

and

\[ dW = m \frac{d\mathbf{v}}{dt} \cdot \left( \frac{d\mathbf{r}}{dt} dt \right). \]

Because

\[ \frac{d\mathbf{r}}{dt} = \mathbf{v}, \]

we see that

\[ dW = m\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} dt = m\mathbf{v} \cdot d\mathbf{v}. \]

And since

\begin{align*}
\frac{1}{2} d(\mathbf{v} \cdot \mathbf{v}) &= \frac{1}{2} \left[ (\mathbf{v} + d\mathbf{v}) \cdot (\mathbf{v} + d\mathbf{v}) - \mathbf{v} \cdot \mathbf{v} \right] \\
&\approx \frac{1}{2} [\mathbf{v} \cdot \mathbf{v} + 2\mathbf{v} \cdot d\mathbf{v} - \mathbf{v} \cdot \mathbf{v}] = \mathbf{v} \cdot d\mathbf{v}
\end{align*}
we conclude
\[ dW = m\vec{v} \cdot d\vec{v} \equiv d\left(\frac{1}{2}m\vec{v}^2\right), \]
or
\[ W_f - W_0 = \frac{1}{2}m\vec{v}^2. \]
That is, the work done is equal to the kinetic energy imparted.

**Harmonic oscillator:**

\[ F_x = -kx \]

\[ m\ddot{x} = -kx \]

let

\[ k = m\Omega^2 \]

then

\[ \ddot{x} + \Omega^2 x = 0. \]

This is the equation of motion for an undamped, undriven ("free") harmonic oscillator.

Such equations can always be integrated immediately using an integrating factor: let

\[ \ddot{x} = f(x); \]
then $\dot{x}$ is an integrating factor, since if

$$f(x) = -\frac{dV}{dx},$$

$$\dot{x}\ddot{x} = \dot{x} f(x)$$

implies

$$\frac{d}{dt} \left( \frac{1}{2} \dot{x}^2 + V(x) \right) = 0,$$

or

$$\frac{1}{2} \dot{x}^2 + V(x) = \text{constant}.$$  

Applying this to the harmonic oscillator equation, we have

$$\frac{d}{dt} \left( \frac{1}{2} \dot{x}^2 + \frac{1}{2} \Omega^2 x^2 \right) = 0$$

or

$$\frac{1}{2} m \ddot{x}^2 + \frac{1}{2} k x^2 = E = \text{constant}.$$  

The latter is just the statement that if a force can be expressed as the (negative) derivative of a function of coordinates alone (such forces are called “conservative”), then the sum of the kinetic energy and the work done in acting against this force is constant in time (“conserved”).

If we include (linear) damping and a driving force we obtain the equation for the driven, damped harmonic oscillator:

$$m \ddot{x} = -kx - 2m\gamma \dot{x} + f(t).$$

The damping force is dissipative, i.e. non-conservative.