PHYS 321 Lecture 6 (9 September 2002)

Hooke’s Law is
\[ F = -kx = -\frac{dV}{dx} \]
corresponding to a potential energy
\[ V(x) = \frac{1}{2}kx^2. \]
Where does this kind of force arise? Matter is stable because the forces between atoms are repulsive at short distances, attractive at long distances, and fall to zero as \( r \to \infty \). That is, the interatomic potentials have a minimum at the equilibrium spacing, as shown to the right. Near the minimum we can expand in Taylor’s series:

\[ V(r) = V(r_0) + (r - r_0) \frac{dV}{dr} \bigg|_{r=r_0} + \frac{1}{2} \frac{d^2V}{dr^2} \bigg|_{r=r_0} (r - r_0)^2 + \ldots \]

Since the first derivative vanishes at a minimum, we see that \( V(r) \) is approximately quadratic for small deviations from equilibrium, meaning that the force approximately obeys Hooke’s Law. Thus when we slightly distort some piece of material—such as a spring—the restoring force tends to be linear in the displacement.

Potentials with minima occur in other situations than springs. For example, a simple pendulum is stable at bottom dead center. The equation for a physical pendulum is
\[ m\ell^2 \ddot{\theta} = -mg \ell \sin \theta = -mg \ell \theta, \]
where we approximate the sine of an angle by the angle, for small angles. In the latter case, pendulum motion is described by the harmonic oscillator equation below.

We now discuss the simple harmonic oscillator. Without damping or an external driving force, the displacement obeys the equation
\[ m \ddot{x} = -kx \]
or, with the definition \( k = m\omega^2 \) we have
\[ x + \omega^2 x = 0 \, . \]

We can always obtain a first integral of any equation of the form
\[ \dot{x} = f(x) \, ; \]
for letting \( \frac{dF}{dx} = f(x) \) we see that
\[ \dot{x}
\left [ x - f(x) \right ] = \frac{d}{dt} \left [ \frac{1}{2} \dot{x}^2 - F(x) \right ] = 0 \, . \]

In the present case,
\[ F(x) = -\frac{1}{2} \omega^2 x^2 \]
so
\[ \frac{1}{2} \dot{x}^2 + \frac{1}{2} \omega^2 x^2 = \frac{E}{m} = \text{constant} \]
We recognize this as the statement that the sum of kinetic and potential energies is conserved.

We can solve this equation in a variety of ways. First, we can use separation of variables: solve for \( \dot{x} \) and write
\[ \dot{x} = \frac{dx}{dt} = \pm \sqrt{\frac{2E}{m} - \omega^2 x^2} \]
or
\[ \frac{dx}{\sqrt{\frac{2E}{m} - \omega^2 x^2}} = \pm dt \, . \]
We can change variables in the latter equation:
\[ \sqrt{\frac{m\omega^2}{2E}} = u \]
leading to
\[ \int \frac{du}{\sqrt{1 - u^2}} = \pm \omega \int dt = \pm \omega t + \phi \, . \]
That is,
\[ x(t) = x_0 \cos(\omega t) + \frac{\dot{x}_0}{\omega} \sin(\omega t) \, . \]
Alternatively, we might have noted that any homogeneous linear differential equation with constant coefficients,
\[ \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_0 x = 0 \, , \]
has the solution
\[ x(t) = A e^{\lambda t} \, . \]
In the present second-order harmonic oscillator equation we find, on substitution,
\[ x + \omega^2 x = \left ( \lambda^2 + \omega^2 \right ) A e^{\lambda t} = 0 \]
or $\lambda = \pm i \omega$, $i = \sqrt{-1}$. Since
\[ e^{\pm i \omega t} = \cos(\omega t) \pm i \sin(\omega t) \]
we see that the general solution consists of sines and cosines.

We can solve the homogeneous equation with (linear) damping,
\[ x + \gamma x + \omega^2 x = 0, \]
the same way. We see that
\[ (\lambda^2 + \gamma \lambda + \omega^2) e^{\lambda t} = 0 \]
or
\[ \lambda = \frac{-\gamma}{2} \pm i \Omega, \quad \Omega = \sqrt{\omega^2 - \gamma^2 / 4}. \]

Then the general solution becomes
\[ x(t) = e^{-\gamma t / 2} \left[ x_0 \cos(\Omega t) + \frac{1}{\Omega} \left( x_0 + \frac{\gamma}{2} x_0 \right) \sin(\Omega t) \right]. \]