PHYS 321 Lecture 7 (11 September 2002)

Solution of the driven, damped harmonic oscillator problem

There are several standard methods for solving ode’s, with or without constant coefficients. First we shall do variation of parameters, then we shall use Heaviside’s operator method.

Variation of parameters

The equation describing the driven, damped harmonic oscillator is

$$x + \gamma \dot{x} + \omega^2 x = \frac{1}{m} f(t)$$

(1)

where $f(t)$ is the driving force, $-\gamma \dot{x}$ the damping force, and $-m\omega^2 x$ the restoring force. Now we have already seen that the general solution of the homogeneous equation (that is, with no driving force) is

$$x(t) = A e^{-\gamma/2} e^{i\Omega t} + B e^{-\gamma/2} e^{-i\Omega t} \equiv Ax_1(t) + Bx_2(t)$$

where $\Omega = \sqrt{\omega^2 - \gamma^2 / 4}$ and $A$ and $B$ are arbitrary constants. The trick of variation of parameters is to let $A$ and $B$ be functions of time, such that

$$A x_1(t) + B x_2(t) = 0.$$  

(2)

Although this condition seems slightly artificial, this choice makes the guess

$$x(t) = A(t) x_1(t) + B(t) x_2(t)$$

as good an approximation to the solution as possible, at the time $t$. It is worth noting that condition 2 above implies that

$$\dot{A} x_1(t) + \dot{B} x_2(t) + A \ddot{x}_1(t) + B \ddot{x}_2(t) = 0.$$

We then substitute into Eq. 1 to get

$$\dot{A} x_1(t) + \dot{B} x_2(t) = \frac{1}{m} f(t)$$

which, with Eq. 2, gives

$$\dot{A} = \frac{x_2(t)}{m W(t)} f(t)$$

$$\dot{B} = -\frac{x_1(t)}{m W(t)} f(t)$$

where $W(t) = x_1(t) x_2(t) - x_1(t) x_2(t)$. We can now solve these equations by integrating with respect to time. The result will depend on the initial conditions we impose—for example, we might want to start the mass with zero displacement, giving

$$A(t) = \int_0^t \frac{x_2(s)}{m W(s)} f(s) \, ds$$

$$B(t) = -\int_0^t \frac{x_1(s)}{m W(s)} f(s) \, ds$$
For the damped harmonic oscillator, with $x_1$ and $x_2$ as given, we have

$$W(t) = 2i\Omega e^{-\gamma t}$$

and

$$x(t) = A(t)x_1(t) + B(t)x_2(t) = \frac{1}{m\Omega} \int_0^t e^{(s-t)\gamma/2} \sin[\Omega(t-s)] f(s) \, ds.$$ 

**Heaviside’s operator method**

This method is equivalent to the Laplace transform. It works only for equations with constant coefficients, like the harmonic oscillator equation, whereas variation of parameters will work for any linear inhomogeneous equation of any degree (assuming one knows all the independent solutions of the homogeneous equation!!).

We replace the differentiation operator $\frac{d}{dt}$ with the symbol $D$ everywhere in a linear differential equation with constant coefficients. This gives, in the harmonic oscillator example,

$$(D^2 + \gamma D + \omega^2)x = \frac{1}{m} f(t).$$

The formal solution is

$$x = (D^2 + \gamma D + \omega^2)^{-1} \frac{1}{m} f$$

so if we know how to evaluate this we are done. The idea is to write the polynomial in $D$ in factored form:

$$D^2 + \gamma D + \omega^2 = \left(D + \frac{\gamma}{2}\right)^2 + \omega^2 = (D - \lambda)(D - \lambda^*)$$

$$\lambda = -\frac{\gamma}{2} + i\Omega$$

where $\Omega$ has the same meaning as before. Now if $D$ were just a number, we could obviously write

$$(D^2 + \gamma D + \omega^2)^{-1} \equiv \frac{1}{2i\Omega} \left( \frac{1}{D + \frac{\gamma}{2} - i\Omega} - \frac{1}{D + \frac{\gamma}{2} + i\Omega} \right);$$

proceeding as if this relation were true, we have to interpret the terms

$$y = \frac{1}{D + \frac{\gamma}{2} - i\Omega} f, \quad y^* = \frac{1}{D + \frac{\gamma}{2} + i\Omega} f.$$ 

We see that if we multiply through by the operator $D + \frac{\gamma}{2} - i\Omega$ we find

$$(D + \frac{\gamma}{2} - i\Omega)y = f$$

which is a first-order linear ode which can be solved by an integrating factor:

$$e^{(\frac{\gamma}{2} - i\Omega)t} (D + \frac{\gamma}{2} - i\Omega)y \equiv \frac{d}{dt} \left(e^{(\frac{\gamma}{2} - i\Omega)t} y(t) \right) = e^{(\frac{\gamma}{2} - i\Omega)t} f(t)$$

or
\[ y(t) = e^{-\frac{t}{\Omega}} \int_0^t e^{\frac{t-s}{\Omega}} f(s) \, ds. \]

Substituting this in the previous equation, we obtain the result we got earlier using variation of parameters:

\[ x(t) = \frac{1}{2\imath m\Omega} \left[ y(t) - y^*(t) \right] = \frac{1}{m\Omega} \int_0^t e^{(s-t)/\gamma} \sin \left[ \Omega(t-s) \right] f(s) \, ds. \]

**Sinusoidal driving force**

Consider a driving force 

\[ f(t) = f_0 \sin \alpha t \]

and find the solution that starts with \( x(t) = 0 \). We keep only the terms that survive for large \( t \):

\[ x(t) \to \frac{f_0}{2m\Omega} \text{Re} \left[ \frac{e^{-\imath t}}{\frac{i}{2} - \imath \left( \Omega + \alpha \right)} - \frac{e^{\imath t}}{\frac{i}{2} - \imath \left( \Omega - \alpha \right)} \right]. \]

We see that as \( \alpha \to \Omega \) one of the terms becomes much larger in magnitude than the other. This phenomenon is called resonance. At resonance the solution is

\[ x(t) = -\frac{f_0}{m\Omega \gamma} \cos \alpha t \]

— that is, in the absence of damping (\( \gamma \to 0 \)) the amplitude of oscillation would become infinite.