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Chapter 6

Angular Momentum

6.1 Commutation Relations

We have already shown that we can define angular momentum operators from the classical definition:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{6.1}$$

with components:

$$L_x = yp_z - zp_y = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \tag{6.2}$$

$$L_y = zp_x - xp_z = \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \tag{6.3}$$

$$L_z = xp_y - yp_x = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \tag{6.4}$$

Derivation of $[L_i, L_j]$:

Using the fundamental commutation relations for coordinates and momenta, namely:

$$[x_i, x_j] = 0, \qquad [p_i, p_j] = 0, \qquad [x_i, p_j] = i\hbar\delta_{ij} \tag{6.5}$$

where $x_1 = x$, $x_2 = y$, $x_3 = z$, etc, we can obtain the commutative properties of angular momentum:

$$[L_x, L_y] = [yp_z - zp_y, L_y] = [yp_z, l_y] - [zp_y, L_y] = y[p_z, L_y] - [zp_y, L_y]$$

$$= y[p_z, zp_x - xp_z] - [zp_y, zp_x - xp_z]$$

$$= y[p_z, zp_x] + [zp_y, xp_z] = yp_x[p_z, z] + xp_y[z, p_z] = i\hbar(xp_y - yp_x)$$

so:

$$[L_x, L_y] = i\hbar L_z \tag{6.6}$$

Similarly, by permuting the indices, x, y, z, we obtain:

$$[L_y, L_z] = i\hbar L_x, \qquad [L_z, L_x] = i\hbar L_y \tag{6.7}$$

Proof that $[L^2, \vec{L}] = 0$:

We can also check the commutation of L^2 with L_z:

$$[L^2, L_z] = [L_x^2, L_z] + [L_y^2, L_z] + [L_z^2, L_z] \tag{6.8}$$

$$= L_x[L_x, L_z] + [L_x, L_z]L_x + L_y[L_y, L_z] + [L_y, L_z]L_y$$

$$= -i\hbar L_x L_y - i\hbar L_y L_x + i\hbar L_y L_x + i\hbar L_x L_y = 0$$

There is nothing special about L_z so we conclude that L^2 commutes with all the components of $\vec{L}$:

$$[L^2, \vec{L}] = 0 \tag{6.9}$$

Proof that $[H, \vec{L}] = 0$:

$$[H, L_z] = [-\frac{\hbar^2}{2m}p^2 + V(r), L_z] = -\frac{\hbar^2}{2m}[p^2, L_z] + [V(r), L_z] \tag{6.10}$$

We have already noted that in spherical coordinates:

$$L_z = -i\hbar\frac{\partial}{\partial\phi} \tag{6.11}$$

Thus $L_z V(r) f = V(r) L_z f$, or L_z commutes with $V(r)$. Thus:

$$[H, L_z] = -\frac{\hbar^2}{2m}[p^2, L_z] \tag{6.12}$$

Working out the components:

$$[p^2, L_z] = [p_x^2 + p_y^2 + p_z^2, xp_y - yp_x] = [p_x^2 + p_y^2, xp_y - yp_x] \tag{6.13}$$

$$= [p_x^2, xp_y] - [p_y^2, yp_x] = [p_x^2, x]p_y - [p_y^2, y]p_x$$

$$= -2i\hbar p_x p_y + 2i\hbar p_y p_x = 0$$

so we obtain:

$$[H, L_z] = 0 \tag{6.14}$$

Since the choice of z axis is arbitrary, it follows that:

$$[H, \vec{L}] = 0 \tag{6.15}$$

Proof that $[H, L^2] = 0$

Since H commutes with each of the components of $\vec{L}$ it also commutes with L^2:

$$HL^2 = H(L_x^2 + L_y^2 + L_z^2) \tag{6.16}$$

and:

$$HL_x^2 = (HL_x)L_x = L_x H L_x = L_x^2 H \tag{6.17}$$

so:

$$[H, L^2] = 0 \tag{6.18}$$

6.1.1 Angular momentum ladder operators

Although the various components of $\vec{L}$ do not commute with each other, the fact that they commute with the Hamiltonian means that if f is an eigenfunction of H so that $Hf = Ef$, then so is $g = L_x f$:

$$Hg = HL_x f = L_x f H = L_x E\ f = EL_x f = Eg \tag{6.19}$$

The eigenvalue is the same. Thus the components of $\vec{L}$ acting on an eigenfunction of H result in another eigenfunction of H with the same eigenvalue. The fact that the component of $\vec{L}$ do not commute with each other but do commute with H and L^2 implies that there must be a degeneracy associated with the eigenvalues of energy and angular momentum resulting in several distinct eigenfunctions. We have obtained this degeneracy by a classical solution of the differential equation.

Linear combinations of the components of L_x and L_y will also commute with H and L^2 but not with L_z. We can define two important combinations of L_x and L_y:

$$L_{\pm} = L_x \pm iL_y \tag{6.20}$$

These obey the commutation relations:

$$[L_z, L_{\pm}] = [L_z, L_x \pm iL_y] = [L_z, L_x] \pm i[L_z, L_y]$$

$$= i\hbar L_y \mp i^2 \hbar L_x = \hbar(\pm L_x + iL_y) = \pm(L_x \pm iL_y)$$

$$[L_z, L_{\pm}] = \pm\hbar L_{\pm} \tag{6.21}$$

$$[L^2, L_{\pm}] = 0 \tag{6.22}$$

$$[L_+, L_-] = [L_x + iL_y, L_x - iL_y] = -i[L_x, L_y] + i[L_y, L_x] = -2i(i\hbar L_z)$$

$$[L_+, L_-] = 2\hbar L_z \tag{6.23}$$

Since these operators are linear combinations of operators that commute with the Hamiltonian, then also commute with H. Therefore, when they act on an eigenfunction the result must be a linear combination of eigenfunctions with the same energy.

Since H, L^2 and L_z all mutually commute, we can find simultaneous eigenfunctions of all three. Let's see what effect $L_\pm$ has on one of these eigenfunctions. We begin by assuming that we don't yet know the eigenvalues of L^2 and L_z. Let the eigenvalue of L^2 be $\lambda\hbar^2$, and let the eigenvalue of L_z be $\mu\hbar$. Let $f_{E_f,\lambda,\mu}$ be an eigenfunction of H, L^2 and L_z. Let $g = L_\pm f$:

$$L^2 g = L^2(L_\pm f) = L_\pm(L^2 f) = \lambda\hbar^2(L_\pm f) = \lambda\hbar^2 g \tag{6.24}$$

$$L_z g = L_z(L_\pm f) = (L_\pm L_z \pm \hbar L_\pm)f = (\mu\hbar \pm \hbar)(L_\pm f) = (\mu \pm 1)\hbar g \tag{6.25}$$

Thus g is an eigenfunction of H, L^2 and L_z, but its eigenvalue for L_z is higher or lower by one unit of $\hbar$ depending on whether we use L_+ or L_-. The origin of the names *raising* and *lowering* operators becomes apparent. It follows that in vector form:

$$\begin{cases} |g_+\rangle \equiv L_+|E_f,\lambda,\mu\rangle = \alpha|E_f,\lambda,(\mu+1)\rangle \\ |g_-\rangle \equiv L_-|E_f,\lambda,\mu\rangle = \beta|E_f,\lambda,(\mu-1)\rangle \end{cases} \tag{6.26}$$

where α and β are undetermined.

6.1.2 Algebraic Derivation of the Eigenvalues for Angular Momentum

We will now derive the eigenvalues of the angular momentum algebraically from the commutation relations. We start fresh and imagine that we have not already solved the angular part of the Schrödinger Equation . The method proceeds as follows:

1. From the fact that the Hermitian operators, H,L^2 and L_z all commute with each other, we know that we can find states that are simultaneously eigenstates of all three.

2. The eigenstates are specified by their energy, and their values for L^2 and L_z. Denote the eigenvalues of L_z by $\mu\hbar$ and the eigenvalues of L^2 by $\lambda\hbar^2$. The use of the $\hbar$ makes μ and λ dimensionless.

3. Use the fact that the operators $L_\pm$ act on the eigenstates to yield new eigenstates of H, L^2, L_z, with the same eigenvalues for H and L^2 but with the eigenvalue for L_z different by $\pm\hbar$.

4. Arguing that the eigenvalue for L_z cannot exceed the total angular momentum, we then obtain the quantization conditions for both the eigenvalues of L_z and L^2.

5. The quantization conditions reproduce what we obtained before, but they will also imply a solution that we eliminated in our previous analysis.

If we apply L_+ successively to one of the eigenfunctions we will eventually reach states for which the z component of angular momentum exceeds the total angular momentum. Hence there must be a maximum value of $\mu \equiv l$ such that:

$$\begin{cases} L_+|E_f, \lambda, l\rangle = 0 \\ L_z|E_f, \lambda, l\rangle = l\hbar|E_f, \lambda, l\rangle \\ L^2|E_f, \lambda, l\rangle = \lambda\hbar^2|E_f, \lambda, l\rangle \end{cases} \tag{6.27}$$

Note:

$$\begin{aligned} L_+L_- = (L_x + iL_y)(L_x - iL_y) = L_x^2 + L_y^2 + i(L_yL_x - L_xL_y) \\ = L^2 - L_z^2 + i(-i\hbar L_z) \end{aligned} \tag{6.28}$$

Therefore:

$$L^2 = L_+L_- + L_z^2 - \hbar L_z = L_-L_+ + L_z^2 + \hbar L_z \tag{6.29}$$

so:

$$\begin{aligned} L^2|E_f, \lambda, l\hbar\rangle = (L_-L_+ + L_z^2 + \hbar L_z)|E_f, \lambda, l\rangle = l^2\hbar^2 + l\hbar^2 \\ = l(l+1)\hbar^2|E_f, \lambda, l\rangle \end{aligned} \tag{6.30}$$

so that:

$$\lambda = l(l+1) \tag{6.31}$$

Similarly there must be a cut-off in the lowering process to prevent us from obtaining states with L_z with negative z components of angular momentum whose magnitude exceeds the total angular momentum. Let the minimum eigenvalue of L_z be $l'\hbar$. The corresponding eigenstate has the properties:

$$\begin{cases} L_-|E_f, \lambda, l'\rangle = 0 \\ L_z|E_f, \lambda, l'\rangle = l'\hbar|E_f, \lambda, l'\rangle \\ L^2|E_f, \lambda, l'\rangle = \lambda\hbar^2|E_f, \lambda, l'\rangle \end{cases} \tag{6.32}$$

Note that it is the same λ as in the previous case. Then, it follows that:

$$L^2|E_f, \lambda, l'\rangle = (L_+L_- + L_z^2 - \hbar L_z)|E_f, \lambda, l'\rangle$$

$$= l'^2\hbar^2 - l'\hbar^2 = l'(l'-1)\hbar^2|E_f, \lambda, l'\rangle \tag{6.33}$$

so that:

$$\lambda = l'(l'-1) \tag{6.34}$$

There are two solutions to the equation $l'(l'-1) = l(l+1)$. One is $l' = l+1$, which is contradictory since the minimum state can't have a higher eigenvalue than the maximum state. The second is $l' = -l$, which is very satisfying from the point of view of the symmetry we expect in the orientation of L_z. Thus the momentum sub-states run in integer steps of $\hbar$ from $-l\hbar$ to $l\hbar$. There are $2l+1$ such states so $2l+1$ must be an integer. This implies that l is either an integer or an integer $+1/2$. We can now label the eigenstates as:

$$|E_f, l, m\rangle \tag{6.35}$$

with:

$$\begin{cases} L_z|E_f, l, m\rangle = m\hbar|E_f, l, m\rangle \\ L^2|E_f, l, m\rangle = l(l+1)\hbar^2|E_f, l, m\rangle \end{cases} \tag{6.36}$$

This derivation of the quantization of angular momentum follows directly from the commutation relations. Nowhere is any specific representation of the operators or the eigenstates used. The commutation relations define the algebra of a system.

6.2 Eigenfunctions for angular momentum

A straightforward, but tedious algebra leads to:

$$L^2 = L_+L_- + L_z^2 + \hbar L_z = -\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{6.37}$$

The operator in brackets is just the spherical part of the Laplacian, which gave the solution:

$$\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_l^m = -l(l+1)Y_l^m \tag{6.38}$$

Thus, it follows that:

$$L^2Y_l^m = l(l+1)]\hbar^2Y_l^m \tag{6.39}$$

where, as we found when we solved the Schrödinger Equation , l is a non-negative integer. The spherical harmonics are also eigenfunctions of L_z:

$$L_zY_l^m = \frac{\hbar}{i}imY_l^m = m\hbar Y_l^m \tag{6.40}$$

6.3 Appendix

Elegant method to determine L^2

There is an elegant and very useful way to do cross products. Define the anti-symmetric symbol ϵ_{ijk} where:

$$\begin{cases} \epsilon_{123} = \epsilon_{213} = \epsilon_{312} = 1 & \\ \epsilon_{321} = \epsilon_{132} = \epsilon_{213} = -1 & \\ \epsilon_{ijk} = 0, & \text{if any two indices are the same} \end{cases} \tag{6.41}$$

The product of two symbols has the important property:

$$\sum_i \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{6.42}$$

Using this symbol the cross product of two vectors can be written as:

$$(\mathbf{A} \times \mathbf{B})_i = \sum_{jk} \epsilon_{ijk} A_j B_k \tag{6.43}$$

where we use the notation that $A_1 = A_x$, $A_2 = A_y$ and $A_3 = A_z$:

$$L^2 = (\vec{r} \times \vec{p}) \cdot (\vec{r} \times \vec{p}) = \sum_i (\vec{r} \times \vec{p})_i (\vec{r} \times \vec{p})_i \tag{6.44}$$

$$(\vec{r} \times \vec{p})_i = \sum_{jk} \epsilon_{ijk} x_j p_k = \sum_{lm} \epsilon_{ilm} x_l p_m \tag{6.45}$$

$$L^2 = \sum_i \left(\sum_{jk} \epsilon_{ijk} x_j p_k \right) \left(\sum_{lm} \epsilon_{ilm} x_l p_m \right) \tag{6.46}$$

$$L^2 = \sum_{ijklm} \epsilon_{ijk}\epsilon_{ilm} x_j p_k x_l p_m \tag{6.47}$$

Summing first over i, which does not appear in the subscripts on x or p:

$$L^2 = \sum_{jklm} (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) x_j p_k x_l p_m \tag{6.48}$$

Next, sum over l and m, taking advantage of the Kronecker δ symbols:

$$= \sum_{jk} (x_j p_k x_j p_k - x_j p_k x_k p_j) \tag{6.49}$$

Using the commutation relations:

$$= \sum_{jk} \left(x_j (x_j p_k - i\hbar\delta_{jk}) p_k - (p_k x_j + i\hbar\delta_{jk}) x_k p_j \right) \tag{6.50}$$

$$= \sum_{jk} \left(x_j (x_j p_k - i\hbar\delta_{jk}) p_k - p_k x_k x_j p_j - i\hbar\delta_{jk} x_k p_j \right) \tag{6.51}$$

$$= \sum_{jk} \left(x_j x_j p_k p_k - p_k x_k x_j p_j \right) - 2i\hbar \sum_j x_j p_j \tag{6.52}$$

In vector notation:

$$L^2 = r^2 p^2 - (\mathbf{p} \cdot \mathbf{r})(\mathbf{r} \cdot \mathbf{p}) - 2i\hbar \mathbf{r} \cdot \mathbf{p} \tag{6.53}$$

Finally:

$$\mathbf{p} \cdot \mathbf{r} = p_x x + p_y y + p_z z = x p_x + y p_y + z p_z - 3i\hbar = \mathbf{r} \cdot \mathbf{p} - 3i\hbar \tag{6.54}$$

so:

$$L^2 = r^2 p^2 - (\mathbf{r} \cdot \mathbf{p})^2 + i\hbar \mathbf{r} \cdot \mathbf{p} \tag{6.55}$$

$$(\mathbf{r} \cdot)(\mathbf{r} \cdot \mathbf{p}) = r p_r r p_r = r(r p_r - i\hbar) p_r = r^2 p_r^2 - i\hbar r p_r \tag{6.56}$$

$$L^2 = r^2 p^2 - r^2 p_r^2 + 2i\hbar r p_r \tag{6.57}$$