Contents

Chapter 7

Spin

7.1 Spin Angular Momentum

In solving the central force problem, we used the differential equation form of the Schrödinger Equation. The requirement that the solutions, which were functions of the space angles θ and ϕ, be invariant when $\phi \to \phi + 2\pi$, led to the integer quantum numbers m and l for the angular part of the solution. We later identified these as the quantum numbers associated with the eigenvalues of the angular momentum differential operators, L^2 and L_z. Using the algebraic method, we have obtained the same quantization conditions from the commutators alone. We also obtained something more, the possibility that l and m could be half-integer (using the common slang). If we take the algebra seriously, we can define another type of angular momentum, but apparently it cannot be represented by functions of space.

A puzzling feature of atomic spectra was discovered by the Dutch physicist, Pieter Zeeman, who studied the emission spectra of atoms in magnetic fields. It was expected that atoms with non-zero orbital angular momentum should have a magnetic moment proportional to the angular momentum. For some arbitrary z axis, the magnetic moment should have only discrete values, μm_z, where m_z takes on integer values from $-l$ to l, and μ is a constant that we shall work out later in the course. For an atom with angular momentum of $l = 1$ one expects $\vec{\mu} \cdot \vec{B}$ to have three possible values, $-\mu B$, 0, μB. For $l = 0$, only 0 is possible, for $l = 2$, there are five values, etc. There are always an odd number of values, with one of them being 0. In an atom the magnetic field should split any given level into an odd number of distinct levels. Transitions between two levels, each of which are split into an odd number of sub levels, should show an odd number of possible energy steps. In many cases, the spacing between the sub-levels is the same for each of the two main levels. If one is split into 5 levels and the other into 3 there will be 15 possible transitions, but with equal splitting of each level we would actually observe only seven distinct energy transitions. In any case the expected splitting of the spectral lines in the presence of a magnetic field is odd. Zeeman observed more often than not that the spectral lines were split into an even number of lines. This puzzling result was called the anomalous Zeeman effect.

In 1922 Stern and Gerlach[1] published the results of what is now a very famous experiment. We will discuss it in more detail later in this chapter. For now we simply outline the basic idea. If a magnetic moment passes through a non-uniform magnetic field with mostly a B_z component there will be a net force deflecting the particle from a straight line path. The deflection will depend on $\vec{\mu} \cdot \vec{B}$. They constructed an apparatus that permitted them to pass a beam of atoms in the y direction through a magnetic field B_z, but non-uniform in the sense that B_z decreased on both sides of $x = 0$. They passed silver atoms through the apparatus expecting to see an odd number of discrete beams emerging from the apparatus if QM was correct, and to see atoms emerging in a smear if classical ideas were correct. They did indeed see discrete beams emerging from the apparatus, but with a new surprise, two beams emerged rather than an odd number. Their experiment demonstrated the quantization effects predicted by QM, but not the expected number of states.

The Stern-Gerlach apparatus is frequently used as a model for the quintessential quantum mechanical measurement device. Systems in an arbitrary state enter the device and emerge sorted into distinct states that are eigenstates of the device (in this case eigenstates of the z component of angular momentum).

The Zeeman effect and the Stern-Gerlach results were explained simultaneously by two young Dutch physicists, George Uhlenbeck and Samuel Goudsmit with the hypothesis that electrons carried an intrinsic angular momentum, which is called **spin**[2]. They hypothesized that the quantum number associated with the spin angular momentum was $l = 1/2$. In this case there are two possible sub-states for the z component, $m = \pm 1/2$. A classical analog for intrinsic angular momentum and orbital angular momentum is provided by the angular momentum of the earth which spins around its own axis while rotating around the sun. A major difference here is that

[1] W. Gerlach and O. Stern, *Zeit. für Phys.*, **9**, 349 (1922)

[2] S.A. Goudsmit and G. Uhlenbeck, *Naturwiss.* **47**, 164 (1925).

the earth's spin angular momentum is easily interpreted as the sum of the angular momenta of its constituents which are rotating around its axis. Its total angular momentum could be calculated by adding the angular momenta of each constituent, which is $m_i\vec{r}_i \times \vec{v}_i$ where $\vec{v}_i$ is the velocity of m_i relative to the sun. As far as we know the electron has no constituents.

We use $\vec{S}$ to denote the intrinsic, or spin angular momentum. The spin of a particle is independent of its spatial wave function. The algebra is the same as for orbital angular momentum:

$$[S_x, S_y] = i\hbar S_z \tag{7.1}$$

$$[S^2, \vec{S}] = 0 \tag{7.2}$$

$$S_\pm = S_x \pm iS_y \tag{7.3}$$

$$[S_z, S_\pm] = \pm\hbar S_\pm \tag{7.4}$$

$$[S^2, S_\pm] = 0 \tag{7.5}$$

7.1.1 Derivation of the constants associated with raising and lowering operators

We can start with the raising and lowering effects of the operators:

$$S_+|sm\rangle = A_{sm}|s, m+1\rangle \tag{7.6}$$

Taking the inner product with $|s, m+1\rangle$:

$$\langle s, m+1|S_+|sm\rangle = A_{sm}\langle s, m+1|s, m+1\rangle \tag{7.7}$$

so:

$$A_{sm} = \langle s, m+1|S_+|sm\rangle \tag{7.8}$$

Next, consider:

$$S_-|s, m+1\rangle = B_{s,m+1}|s, m\rangle \tag{7.9}$$

Taking the inner product with $|s, m\rangle$:

$$\langle s, m|S_-|s, m+1\rangle = B_{s,m+1}\langle s, m|s, m\rangle \tag{7.10}$$

so:

$$B_{s,m+1} = \langle s, m|S_-|s, m+1\rangle \tag{7.11}$$

Taking the complex conjugate of Eq. 7.8:

$$A^*_{sm} = \langle s, m|S^\dagger_+|sm+1\rangle \tag{7.12}$$

But $S_+^\dagger = S_-$, so:

$$A_{sm}^* = \langle s, m|S_-|sm+1\rangle = B_{sm+1} \tag{7.13}$$

Now:

$$S^2 = S_+S_- + S_z^2 - \hbar S_z = S_-S_+ + S_z^2 + \hbar S_z \tag{7.14}$$

so:

$$S^2 - S_z^2 - \hbar S_z = S_-S_+ \tag{7.15}$$

Applying the left hand side of Eq. 7.15 to an eigenfunction of S^2 and S_z:

$$(S^2 - S_z^2 - \hbar S_z)|sm\rangle = \left(s(s+1) - m^2 - m\right)\hbar^2|sm\rangle \tag{7.16}$$

$$= ((s(s+1) - m(m+1))\,\hbar^2|sm\rangle \tag{7.17}$$

Applying the right hand side of Eq. 7.15 to the same eigenvector:

$$S_-S_+|sm\rangle = S_-(A_{sm}|sm+1\rangle) = A_{sm}B_{sm+1}|sm\rangle = |A_{sm}|^2|sm\rangle \tag{7.18}$$

so:

$$|A_{sm}|^2 = (s(s+1) - m(m+1))\,\hbar^2 \tag{7.19}$$

Since $s \geq m$ we can take A_{sm} to be real:

$$A_{sm} = \sqrt{s(s+1) - m(m+1)}\ \hbar \tag{7.20}$$

$$B_{sm} = A_{sm-1} = \sqrt{s(s+1) - m(m-1)}\ \hbar \tag{7.21}$$

We see that when $m = s$, $A_m = 0$ and when $m = -s$, $B_m = 0$. Since m changes in steps of 1, there are $2s+1$ values of m. The spin angular momentum can take on both integer and half-integer values, i.e.:

$$s = 0,\ \pm\frac{1}{2},\ \ \pm 1,\ \ \pm\frac{3}{2},\ \ \cdots \tag{7.22}$$

The intrinsic spin of elementary particles is characteristic of the particle. According to the standard model, the world is made of three lepton families, the electron and the electron neutrino, the muon and muon neutrino, and the tau and the tau neutrino. The families seem to be identical except for their family id and their mass.These particles interact through the weak and the electromagnetic force. The strongly interacting particles are three families of quarks, which in the order of increasing mass are the up and down, the strange and charm and the top and bottom. The quarks can interact through the strong (or color) force as well as the electromagnetic and weak. The forces are brought about by the exchange of field quanta, called bosons. The electromagnetic force is mediated by exchange of the neutral and massless photon. The weak force is mediated by exchange of very massive particles the $W^{\pm}$ and the neutral Z particle. The quarks carry in addition to electric charge, another sort of charge called color, and the strong force is mediated by exchange of presumably massless gluons, which are not only absorbed and emitted by particles carrying color charge, but carry color themselves. The quarks and leptons all have intrinsic angular momentum, or spin, of $1/2\ \hbar$. The field bosons all carry spin of 1. Protons and neutrons are in a sense composites of three quarks. They have a net spin of 1/2. Excited states of the proton and neutron have spins of 1/2, 3/2, etc. Mesons, such as the pi meson, the rho meson, and the omega meson, are made of quark-antiquark pairs, and have integer spin, generally 0 or 1. How gravity will fit into this picture is not clear, but if it can also be represented by exchange of field quanta, they would have spin 2.

7.2 Quantum mechanics of spin 1/2 systems

Since the spin of a system is intrinsic it can have no dependence on the spatial extent of the system. Thus, we cannot expect to represent the spin in terms of a function of coordinates. How do we represent the spin state of a particle? An answer to this question provides us with an illustration of the major features of quantum mechanic using a very simple system.

If the total spin is 1/2 there are two sub-states, $m = \pm 1/2$. If we ignore the spatial properties of the quantum state and concentrate on its spin properties, we have a very simple quantum system with only two states. These states have no spatial properties, so that we represent them in terms of the abstract vectors of linear algebra. The states are specified by $s = 1/2$, and $m = \pm 1/2$:

$$\chi_+ = |\frac{1}{2}, \frac{1}{2}\rangle, \qquad \chi_- = |\frac{1}{2}, -\frac{1}{2}\rangle \tag{7.23}$$

Any spin state is a linear combination of these two eigenstates of S, and S_z, so that an arbitrary spin state, χ, has the form:

$$\chi = a\chi_+ + b\chi_- \tag{7.24}$$

Using a matrix notation:

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_+^\dagger = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{7.25}$$

$$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \chi_-^\dagger = \begin{pmatrix} 0 & 1 \end{pmatrix} \tag{7.26}$$

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} \tag{7.27}$$

To normalize the state we require that:

$$|a|^2 + |b^2| = 1 \tag{7.28}$$

This condition can be achieved quite generally by setting:

$$a = \cos\left(\frac{\alpha}{2}\right), \qquad b = e^{i\phi}\sin\left(\frac{\alpha}{2}\right) \tag{7.29}$$

where α and ϕ are arbitrary.

The expectation value of a spin operator in this state is:

$$\langle \mathcal{O} \rangle = \langle \chi|\mathcal{O}|\chi\rangle = \left(a^*\langle\chi_+| + b^*\langle\chi_-|\right)\left(a\mathcal{O}|\chi_+\rangle + b\mathcal{O}|\chi_-\rangle\right)$$

$$= |a|^2\langle\chi_+|\mathcal{O}|\chi_+\rangle + |b|^2\langle\chi_-|\mathcal{O}|\chi_-\rangle + a^*b\langle\chi_+|\mathcal{O}|\chi_-\rangle + ab^*\langle\chi_-|\mathcal{O}|\chi_+\rangle \tag{7.30}$$

$$\langle \mathcal{O} \rangle = \cos^2\left(\frac{\alpha}{2}\right)\langle\chi_+|\mathcal{O}|\chi_+\rangle + \sin^2\left(\frac{\alpha}{2}\right)\langle\chi_-|\mathcal{O}|\chi_-\rangle$$

$$+\frac{1}{2}\sin(\alpha)e^{i\phi}\langle\chi_+|\mathcal{O}|\chi_-\rangle + \frac{1}{2}\sin(\alpha)e^{-i\phi}\langle\chi_-|\mathcal{O}|\chi_+\rangle \tag{7.31}$$

If the operator is Hermitian, $\langle\chi_-|\mathcal{O}|\chi_+\rangle = \langle\chi_+|\mathcal{O}|\chi_-\rangle^*$, so that:

$$\langle \mathcal{O} \rangle = \cos^2\left(\frac{\alpha}{2}\right)\langle\chi_+|\mathcal{O}|\chi_+\rangle + \sin^2\left(\frac{\alpha}{2}\right)\langle\chi_-|\mathcal{O}|\chi_-\rangle + \sin(\alpha)\Re\{e^{i\phi}\langle\chi_+|\mathcal{O}|\chi_-\rangle\} \tag{7.32}$$

which demonstrates that the expectation value is real. If the basis states $\chi_\pm$ are eigenstates of the operator $\mathcal{O}$, such that:

$$\begin{cases} \mathcal{O}|\chi_+\rangle = \lambda_1|\chi_-\rangle; \\ \mathcal{O}|\chi_-\rangle = \lambda_2|\chi_-\rangle \end{cases} \tag{7.33}$$

then:

$$\langle \mathcal{O} \rangle = \cos^2\left(\frac{\alpha}{2}\right)\lambda_1 + \sin^2\left(\frac{\alpha}{2}\right)\lambda_2 \tag{7.34}$$

For example, we can verify the expectation value of S^2, for which:

$$\lambda_1 = \lambda_2 = s(s+1)\hbar^2 \tag{7.35}$$

$$\langle S^2\rangle = \left[\cos^2\left(\frac{\alpha}{2}\right) + \sin^2\left(\frac{\alpha}{2}\right)\right] s(s+1)\hbar^2 = s(s+1)\hbar^2 = \frac{3}{4}\hbar^2 \tag{7.36}$$

As another example, we can obtain $\langle S_z\rangle$. For this operator:

$$\begin{cases} \lambda_1 = \frac{\hbar}{2} : \\ \lambda_2 = -\frac{\hbar}{2} \end{cases} \tag{7.37}$$

and:

$$\langle S_z\rangle = \left[\cos^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\alpha}{2}\right)\right] \frac{\hbar}{2} = \frac{\hbar}{2}\cos\alpha \tag{7.38}$$

The expectation value of S_z ranges from $-\hbar/2$ to $\hbar/2$.

We can also calculate the expectation values of S_x and S_y. For these operators the basis states are not eigenstates. It is convenient to write $S_x = \frac{1}{2}(S_+ + S_-)$, and $S_y = -\frac{i}{2}(S_+ - S_-)$. Our calculation will need the results of $S_\pm$ operating on the basis states:

$$S_+\chi_- = \hbar\sqrt{s(s+1) - m(m+1)}\ \chi_+ = \hbar\sqrt{3/4 + 1/4}\,\chi_+ = \hbar\chi_+ \tag{7.39}$$

$$S_+\chi_+ = 0 \tag{7.40}$$

$$S_-\chi_+ = \hbar\sqrt{s(s+1) - m(m-1)}\ \chi_- = \hbar\chi_- \tag{7.41}$$

$$S_-\chi_- = 0 \tag{7.42}$$

With these relations at hand, the expectation values of S_x and S_y can be obtained rather easily:

$$\begin{aligned}
\langle S_x\rangle = \cos^2\left(\frac{\alpha}{2}\right)\frac{1}{2}\langle\chi_+|S_+ + S_-|\chi_+\rangle + \sin^2\left(\frac{\alpha}{2}\right)\frac{1}{2}\langle\chi_-|S_+ + S_-|\chi_-\rangle \\
+\frac{1}{2}\sin(\alpha)e^{i\phi}\frac{1}{2}\langle\chi_+|S_+ + S_-|\chi_-\rangle + \frac{1}{2}\sin(\alpha)e^{-i\phi}\frac{1}{2}\langle\chi_-|S_+ + S_-|\chi_+\rangle \\
= +\frac{1}{2}\sin(\alpha)e^{i\phi}\frac{1}{2}\hbar + \frac{1}{2}\sin(\alpha)e^{-i\phi}\frac{1}{2}\hbar \\
= \frac{\hbar}{2}\sin(\alpha)\frac{1}{2}\left(e^{i\phi} + e^{-i\phi}\right)
\end{aligned}$$

$$\langle S_x\rangle == \frac{\hbar}{2}\sin(\alpha)\cos(\phi) \tag{7.43}$$

Similarly, one can show that:

$$\langle S_y\rangle = \frac{\hbar}{2}\sin(\alpha)\sin(\phi) \tag{7.44}$$

With the z axis fixed, we can choose a set of x and y axes that make $\phi = 0$. Then $\langle S_x\rangle$ ranges from $-\hbar/2$ to $\hbar/2$, and $\langle S_y\rangle = 0$. At the two limits of this range the state is apparently an eigenstate of $\langle S_x\rangle$.

7.2.1 Eigenvalues and eigenvectors of S_x

Choosing $\sin\phi = 0$, and $\sin\alpha = 1$, $\langle S_x \rangle = \hbar/2$ we expect to obtain the positive eigenstate of S_x. If $\sin\alpha = 1$, then $\alpha = \pi/2$. In this case, $\cos(\alpha/2) = \sin(\alpha/2) = \frac{1}{\sqrt{2}}$, so that

$$\psi_+ = \frac{1}{\sqrt{2}}\left(\chi_+ + \chi_-\right) \tag{7.45}$$

Operating on this state with S_x:

$$S_x\psi_+ = \frac{1}{2}(S_+ + S_-)\frac{1}{\sqrt{2}}(\chi_+ + \chi_-) = \frac{\hbar}{2}\frac{1}{\sqrt{2}}(\chi_- + \chi_+) = \frac{\hbar}{2}\psi_+ \tag{7.46}$$

For $\sin(\alpha) = -1$, $\alpha = -\pi/2$, so that $\cos(\alpha/2) = -\sin(\alpha/2) = \frac{1}{\sqrt{2}}$, and:

$$\psi_- = \frac{1}{\sqrt{2}}(\chi_+ - \chi_-) \tag{7.47}$$

Direct application of S_x to this state verifies that it is an eigenstate of S_x with eigenvalue $\hbar/2$.

We can express the z eigenvectors as linear combinations of the x eigenvectors:

$$\chi_+ = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-) \tag{7.48}$$

$$\chi_- = \frac{1}{\sqrt{2}}(\psi_+ - \psi_-) \tag{7.49}$$

An arbitrary spin state can be expressed in terms of either set of eigenstates:

$$\chi = a\chi_+ + b\chi_- = \frac{1}{\sqrt{2}}\left[(a+b)\psi_+ + (a-b)\psi_-\right] \tag{7.50}$$

The probability of measuring $S_x = \hbar/2$ is $|a+b|^2$ and the probability for measuring $S_x = -\hbar/2$ is $|a-b|^2$.

7.2.2 Matrix Representation of Operators in Spin 1/2 Space

With only two basis states, operators can be represented by 2×2 matrices. All such matrices can be written as a linear sum of four independent basis matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{7.51}$$

All 2×2 matrices are linear combinations of these four basis matrices. However, certain linearly independent combinations of them are more commonly used as the basis set. Since:

$$S^2\chi_\pm = s(s+1)\hbar^2\chi_\pm = \frac{3}{4}\hbar^2\chi_\pm \tag{7.52}$$

so:

$$S^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{3}{4}\hbar^2\mathbf{I} \tag{7.53}$$

Also:

$$S_z\chi_+ = \frac{1}{2}\hbar\chi_+ \tag{7.54}$$

$$S_z\chi_- = -\frac{1}{2}\hbar\chi_+ \tag{7.55}$$

it follows that:

$$S_z = \frac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \frac{\hbar}{2}\sigma_z \tag{7.56}$$

From the properties of the raising and lowering operators, it follows that:

$$S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{7.57}$$

$$S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{7.58}$$

The matrices, $\mathbf{I}$, S_z, S_+, and S_- are linearly independent and also can be used as a complete set of basis matrices.

We can easily obtain the matrix representation of S_x and S_y:

$$S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \frac{\hbar}{2}\sigma_x \tag{7.59}$$

$$S_y = \frac{i}{2}(S_- - S_+) = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \equiv \frac{\hbar}{2}\sigma_y \tag{7.60}$$

The three Pauli matrices, σ_x, σ_y, σ_z along with the identity matrix I also form a linearly independent set of 2×2 matrices, and they are in fact the most commonly used basis matrices for spin 1/2. To summarize:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_\mathbf{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\mathbf{y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_\mathbf{z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.61}$$

7.3 The electron in a magnetic field

We will now examine in more detail how we can observe the spin of a particle. Spinning charged particles carry a magnetic moment, just as a current loop has a magnetic moment. For a circular current loop the magnetic moment is perpendicular to the plane of the loop. If the current consists of moving positive charges, the angular momentum is perpendicular to the loop and in the direction of the magnetic moment. In fact, one can show that the magnetic moment of a current loop is proportional to the orbital angular momentum of the flowing charges. Consider a plane circular loop in the $x-y$ plane of radius a carrying a charge q and a mass m. Let the loop rotate around the z axis perpendicular to its plane and passing through its center. Let the angular velocity be ω. Its angular momentum is:

$$\vec{L} = \vec{r} \times \vec{p} = m\omega a^2 \hat{k} \tag{7.62}$$

The current associated with the rotating loop is $I = q/T = q/(2\pi/\omega) = q\omega/(2\pi)$. From electromagnetic theory, a planar loop with normal vector $\hat{k}$ has a magnetic dipole of:

$$\vec{\mu} = IA\hat{k} = I(\pi a^2)\hat{k} = \frac{1}{2}q\omega a^2\hat{k} = \frac{q}{2m}\vec{L} = \gamma\vec{L} \tag{7.63}$$

where:

$$\gamma \equiv \frac{q}{2m} \tag{7.64}$$

The result is the same for a rotating sphere provided the ratio of charge density to mass density is the same everywhere in the rotating object. The factor γ is called the gyromagnetic ratio.

For the electron, $\vec{L} = \vec{S}$, so we expect classically:

$$\vec{\mu} = \gamma\vec{S} \tag{7.65}$$

Dirac's relativistic theory of the electron predicts that γ is twice the classical value. Since the charge of an electron is negative, the magnetic moment is opposite to the spin:

$$\gamma = -\frac{e}{2m} \tag{7.66}$$

For an electron at rest in a magnetic field, the energy of a magnetic dipole in a magnetic field provides the Hamiltonian:

$$H = -\vec{\mu} \cdot \vec{B} = -\gamma\vec{B} \cdot \vec{S} \tag{7.67}$$

Consider a static field in the z direction, (the field defines a preferred direction):

$$\vec{B} = B_0 \hat{k} \tag{7.68}$$

$$H = -\gamma B_0 S_z = -\frac{\gamma B_0 \hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \hbar\omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.69}$$

where I have defined a positive frequency:

$$\omega = -\frac{\gamma B_0 \hbar}{2} = \frac{e B_0 \hbar}{4m} \tag{7.70}$$

The eigenstates of the Hamiltonian are clearly the eigenstates of S_z, namely χ_+ and χ_-:

$$H\chi_+ = \hbar\omega\chi_+ = E_1\chi_+ \tag{7.71}$$

$$H\chi_- = -\hbar\omega\chi_- = E_0\chi_- \tag{7.72}$$

The ground state has the z component of the spin aligned opposite to the magnetic field, which means it has the magnetic moment aligned with the magnetic field, just as for for classical magnetic dipoles.

Let us now consider the time development of an arbitrary state:

$$\chi(t) = a(t)\chi_+ + b(t)\chi_- \tag{7.73}$$

The time dependence appears in the coefficients of the two eigenstates. The relative proportion of each eigenstate in the wave function is time-dependent. The Schrödinger equation is:

$$H\chi = i\hbar\frac{\partial \chi}{\partial t} \tag{7.74}$$

$$\hbar\omega a\chi_+ - \hbar\omega b\chi_- = i\hbar\left[\frac{da}{dt}\chi_+ + \frac{db}{dt}\chi_-\right] \tag{7.75}$$

Since the eigenstates are orthogonal:

$$\frac{da}{dt} = -i\omega a, \qquad \frac{db}{dt} = i\omega b \tag{7.76}$$

The solutions are:

$$a = a_0 e^{-i\omega t}, \qquad b = b_0 e^{i\omega t} \tag{7.77}$$

where the general time-dependent solution for the spin state of the electron is:

$$\chi(t) = a_0 e^{-i\omega t}\chi_+ + b_0 e^{i\omega t}\chi_- \tag{7.78}$$

with the normalization condition:

$$|a_0|^2 + |b_0|^2 = 1 \tag{7.79}$$

As before, we can write:

$$a_0 = \cos(\alpha/2) \tag{7.80}$$

$$b_0 = \sin(\alpha/2) \tag{7.81}$$

so that:

$$\chi(t) = \begin{pmatrix} \cos(\alpha/2)e^{-i\omega t} \\ \sin(\alpha/2)e^{i\omega t} \end{pmatrix} \tag{7.82}$$

Using our general formula for the expectation value of an operator in an arbitrary spin sate, we find:

$$\langle S_z \rangle = \chi(t)^\dagger S_z \chi(t) = \left(\cos^2(\alpha/2) - \sin^2(\alpha/2)\right)\frac{\hbar}{2} = \frac{\hbar}{2}\cos\alpha \tag{7.83}$$

which is independent of time.

The expectation value of S_x is:

$$\langle S_x \rangle = \frac{\hbar}{2}\sin(\alpha)\cos(2\omega t) \tag{7.84}$$

The expectation value of S_y is:

$$\langle S_y \rangle = \frac{\hbar}{2}\sin(\alpha)\sin(2\omega t) \tag{7.85}$$

We see that:

$$\left[\langle S_x \rangle^2 + \langle S_y \rangle^2\right]^2 = \frac{\hbar}{2}\sin(\alpha) \tag{7.86}$$

The spinning electron precesses around the direction defined by the magnetic field.

Electron initially quantized along the x axis.

If the electron is initially in the positive eigenstate ψ_+ of S_x such that:

$$S_x\psi_+ = \frac{\hbar}{2}\psi_+ \tag{7.87}$$

then $a = b = 1/\sqrt{2}$, and $\alpha = \pi/2$:

$$\chi(t) = \frac{1}{\sqrt{2}}\left[e^{-i\omega t}\chi_+ + e^{i\omega t}\chi_-\right] \tag{7.88}$$

The expectation values are:

$$\langle S_z \rangle = 0, \qquad \langle S_x \rangle = \frac{\hbar}{2}\cos(2\omega t), \qquad \langle S_y \rangle = \frac{\hbar}{2}\sin(2\omega t) \tag{7.89}$$

7.3.1 Stern-Gerlach Experiment

In an inhomogeneous magnetic field, there is a net force on a magnetic dipole:

$$\mathbf{F} = \nabla \left(\mu \cdot \mathbf{B} \right) \tag{7.90}$$

If a beam of particles with a particular spin orientation in the z direction travels in the y direction through an inhomogeneous field directed primarily in the z direction, there will be a force in the z direction. To see this, let:

$$\mathbf{B} = -\alpha x \hat{\imath} + (B_0 + \alpha z)\hat{k} \tag{7.91}$$

where $\alpha z << B_0$. This field satisfies Maxwell's equations:

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = 0 \tag{7.92}$$

Then:

$$\mu \cdot \mathbf{B} = -\alpha x \mu_x + \alpha(B_0 + \alpha z)\mu_z = \gamma \left[-\alpha x S_x + \alpha(B_0 + \alpha z) S_z \right] \tag{7.93}$$

The force is:

$$\mathbf{F} = \gamma\alpha(-S_x \hat{\imath} + S_z \hat{k}) \tag{7.94}$$

Because of the rapid Larmor precession around the z direction, the net force in the x direction is 0. We are left with:

$$F_z = \gamma\alpha S_z \tag{7.95}$$

The beam is bent up or down depending on the sign of S_z. Classically, we would expect a smear, but the quantization of the spin angular momentum gives 2s+1 individual beams if the initial beam is an unpolarized mixture. One can insert blockers to eliminate all but one of the beams, and use another magnet to undo the deflection, ending up with a beam of completely polarized particles.

7.4 Addition of Angular Momentum

Most quantum mechanical systems consist of more than one particle. The hydrogen atom, for instance has a proton and an electron. The other atoms have even more particles. The deuterium atom has a nucleus of a proton and a neutron bound to each other and an electron to form the atom. Helium has a nucleus consisting of two protons and two neutrons, and its atomic properties are carried by two electrons. Things quickly become more complicated.

It is especially important to note that protons, neutrons and electrons all have intrinsic spin of $1/2$. What are the possible values for the spin of a deuteron, the helium nucleus, or the hydrogen atom, for example?

For a system of two spin $1/2$ particles, if we ignore the spatial part of the state function, we still must consider the way the two particles combine to give a system of some total angular momentum. We expect the ground state is most likely in a state of 0 orbital angular momentum. From now on, we will follow the standard convention that uses L to describe orbital angular momentum, S to describe spin angular momentum, and J to describe total angular momentum. The quantum numbers for L are integers, the quantum numbers of the other two satisfy the requirement that $2s+1$ and $2j+1$ are integers.

Classically the total angular momentum of two particles of angular momentum $\vec{J}_1$ and $\vec{J}_2$ is:

$$\vec{J} = \vec{J}_1 + \vec{J}_2 \tag{7.96}$$

$$J_x = J_{1x} + J_{2x}, \qquad J_y = J_{1y} + J_{2y}, \qquad J_z = J_{1z} + J_{2z} \tag{7.97}$$

$$J^2 = J_1^2 + J_2^2 + 2\vec{J}_1 \cdot \vec{J}_2 \tag{7.98}$$

$$J_{max}^2 = (J_1 + J_2)^2, J_{min}^2 = (J_1 - J_2)^2 \tag{7.99}$$

so

$$|J_1 - J_2| \leq J \leq J_1 + J_2 \tag{7.100}$$

These two limiting cases correspond to angular momenta in opposite directions or in the same direction. Quantization of angular momentum in quantum systems alters this picture slightly. The total angular momentum must satisfy the condition that $\langle J^2 \rangle = j(j+1)\hbar^2$ where $2j$ is an integer. The z component of the total angular momentum must range from $-j\hbar$ to $j\hbar$. The result is that the total angular momentum quantum number can range from $j = |j_1 - j_2|$ to $j = j_1 + j_2$. For each value of j there are $2j+1$ sub-states each with a different z quantum number, m_j.

For two particles with spin, ignoring the spatial part of the wave function, or by assuming their orbital angular momentum is 0, the total angular momentum is obtained from the operator for the total spin, which is:

$$\vec{S} = \vec{S}^{(1)} + \vec{S}^{(2)} \tag{7.101}$$

For a system of two non-interacting particles, the Hamiltonian is:

$$H = H_1(\mathbf{r_1}) + H_2(\mathbf{r_2}) \tag{7.102}$$

where H_1 depends only on the coordinates of particle 1 and H_2 depends only on the coordinates of particle 2. This Hamiltonian is separable in the two sets of coordinates so we write the two-particle wave function as a product of the single particle wave functions:

$$\psi = \psi_1(1)\psi_2(2) \tag{7.103}$$

For this state:

$$H\psi = [H_1\psi_1(1)]\,\psi_2(2) + \psi_1\,[H_2\psi_2] = i\hbar\left[\frac{\partial\psi_1}{\partial t}\psi_2 + \psi_1\frac{\partial\psi_2}{\partial t}\right] = i\hbar\frac{\partial\psi}{\partial t} \tag{7.104}$$

The same property holds for the combination of spin and angular momentum of a single particle, if the spin and angular momentum do not couple. Expressing the spatial state by $\phi(\vec{r})$ and the spin state by χ, the total wave function is:

$$\psi = \phi\chi \tag{7.105}$$

To combine two particles at rest in spin states, χ_1 and χ_2:

$$\psi = \chi_1(1)\chi_2(2) \tag{7.106}$$

The total z component of angular momentum is:

$$S_z\psi = (S_z^{(1)}+S_z^{(2)})\chi_1(1)\chi_2(2) = [S_z^{(1)}\chi_1(1)]\chi_2(2)+[S_z^{(2)}\chi_2(2)]\chi_1(1) \tag{7.107}$$

The expectation value is:

$$\langle S_z\rangle = \langle S_z\rangle_1 + \langle S_z\rangle_2 \tag{7.108}$$

If each particle is in an eigenstate of S_z:

$$\langle S_z\rangle = (m_1 + m_2)\hbar \tag{7.109}$$

Let us concentrate on the combination of two spin 1/2 particles such as the proton and the electron in the hydrogen atom. If both particles are in the $m = 1/2$ eigenstate of s_z, we can write their joint state vector as:

$$\psi = \chi_+^{(1)}\chi_+^{(2)} \tag{7.110}$$

If we apply the z component of the total spin S_z to this state:

$$S_z\psi = (\hbar/2 + \hbar/2)\psi = \hbar\psi \tag{7.111}$$

so that this is an eigenstate of the total spin, with eigenvalue $m\hbar = \hbar$. Let's apply the raising operator to this state:

$$S_+ = (s_+^{(1)} + s_+^{(2)})\chi_+^{(1)}\chi_+^{(2)} = 0 \tag{7.112}$$

There is no state with a larger value of the z-component of angular momentum, so we can infer that the total angular momentum for this state is $s = 1$. More directly:

$$S^2 = (S^{(1)})^2 + (S^{(2)})^2 + 2\vec{S}^{(1)} \cdot \vec{S}^{(2)} \tag{7.113}$$

We make use of the relations:

$$s_x\chi_+ = \frac{\hbar}{2}\chi_-, \qquad s_y\chi_+ = i\frac{\hbar}{2}\chi_-, \qquad s_z\chi_+ = \frac{\hbar}{2}\chi_+ \tag{7.114}$$

to obtain:

$$\vec{S}^{(1)} \cdot \vec{S}^{(2)}\chi_+^{(1)}\chi_+^{(2)} = (S_x^{(1)}S_x^{(2)} + S_y^{(1)}S_y^{(2)} + S_z^{(1)}S_z^{(2)})\chi_+^{(1)}\chi_+^{(2)} = \frac{\hbar^2}{4}\chi_+^{(1)}\chi_+^{(2)} \tag{7.115}$$

and:

$$S^2\chi_+^{(1)}\chi_+^{(2)} = [\frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2)]\chi_+^{(1)}\chi_+^{(2)} + 2\frac{\hbar^2}{4}\chi_+^{(1)}\chi_+^{(2)} = 2\hbar^2\chi_+^{(1)}\chi_+^{(2)} \tag{7.116}$$

which is the expected result for $S = 1$.

Next consider the state:

$$\psi = \chi_-^{(1)}\chi_-^{(2)} \tag{7.117}$$

Applying the operator S_z:

$$S_z\psi = \left(-\frac{\hbar}{2} - \frac{\hbar}{2}\right)\psi = -\hbar\psi \tag{7.118}$$

so this is an eigenstate of S_z with $m = -1$. It is straightforward to show that $L_-\psi = 0$, so there is no lower eigenstate. If we apply S^2 we find that it is also an eigenstate of $S = 1$. The two states we have found have the z component of angular momentum separated by $2\hbar$. We expect that since the z component of angular momentum changes in steps of $\hbar$ there should be a state with 0 z component of angular momentum. We can use the lowering operator to obtain it from the state with $m = 1$:

$$S_- = S_-^{(1)} + S_-^{(2)} \tag{7.119}$$

Applying this to $|s=1, m=1\rangle$:

$$\left(S_-^{(1)} + S_-^{(2)}\right) \chi_+^{(1)} \chi_+^{(2)} = \hbar(\chi_-^{(1)} \chi_+^{(2)} + \chi_+^{(1)} \chi_-^{(2)}) \tag{7.120}$$

If we normalize the lowered state we obtain:

$$|s=1, m=0\rangle = \frac{1}{\sqrt{2}} \left(\chi_-^{(1)} \chi_+^{(2)} + \chi_+^{(1)} \chi_-^{(2)}\right) \tag{7.121}$$

Finally, we can form a combination of this sort that is orthogonal to the three states we have already found:

$$\psi' = \frac{1}{\sqrt{2}} \left(\chi_-^{(1)} \chi_+^{(2)} - \chi_+^{(1)} \chi_-^{(2)}\right) \tag{7.122}$$

This state has a total angular momentum and a z-component of 0. We can summarize the four states we have found as:

$$l = 1: \quad \begin{cases} \psi_{11} = \chi_+^{(1)} \chi_+^{(2)} \\ \psi_{10} = \frac{1}{\sqrt{2}} \left(\chi_-^{(1)} \chi_+^{(2)} + \chi_+^{(1)} \chi_-^{(2)}\right) \\ \psi_{1-1} = \chi_-^{(1)} \chi_-^{(2)} \end{cases} \tag{7.123}$$

and:

$$l = 0: \quad \psi_{00} = \frac{1}{\sqrt{2}} \left(\chi_-^{(1)} \chi_+^{(2)} - \chi_+^{(1)} \chi_-^{(2)}\right) \tag{7.124}$$

We see that two spin 1/2 particles can have a total spin of 1 or 0. Notice also that the states with total spin 1 do not change if particle 1 and 2 are exchanged, but the state with total spin 0 changes sign under this interchange.

7.4.1 Combining spin and orbital angular momentum

Similar algebraic procedures can be used to combine orbital and spin angular momentum into the total angular momentum, which is the vector sum:

$$\vec{J} = \vec{L} + \vec{S} \tag{7.125}$$

with z component:

$$J_z = L_z + S_z \tag{7.126}$$

The possible values of total angular momentum j range in integer steps from $|l-s|$ to $l+s$. Thus a particle with spin 1/2 and orbital angular momentum l will have a two possible values of total angular momentum, $j = l \pm \frac{1}{2}$. The magnetic sub-states of total angular momentum j range from $m = -j$ to $m = j$ in steps of 1. Of course, these angular momenta are always in units of $\hbar$.

$s = 1/2$ and Orbital angular momentum, l.

As an example what are the possible total angular momentum states for an electron with spin 1/2 in a spatial state with orbital angular momentum, l. We are looking for states with total angular momentum $j = l + 1/2$ and $j = l - 1/2$. Let ϕ_m^l represent the spatial state for a particle of total orbital angular momentum, l, and z component, m. Start by forming the states of maximum and minimum z component of total angular momentum. Clearly the maximum z component is $(l + 1/2)\hbar$ and the minimum is $-(l + 1/2)\hbar$. For these two states it is clear that the total angular momentum quantum number must be $j = l + 1/2$. The two states are:

$$\Phi_{l+1/2}^{l+1/2} = \phi_l^l \chi_+ \tag{7.127}$$

$$\Phi_{-(l+1/2)}^{l+1/2} = \phi_{-l}^l \chi_- \tag{7.128}$$

These are normalized and have $J_z \Phi = \pm(l + 1/2)\hbar\Phi$.

By successive applications of the the lowering operator to the state of highest J_z or the raising operator to the state of lowest J_z we can construct all the intermediate states for $j = l + 1/2$. After completing these steps, we can then examine the state for $j = l + 1/2, m_j = l - 1/2$ and form a new orthogonal linear combination of the states with the same z component of spin. Since it is orthogonal it must belong to a different value of the total angular momentum. The only possibility is $j = l - 1/2$. We can then apply the lowering operator to this state to get all the states with $j = l - 1/2$.

To see how this works consider $l = 1$. The states with maximum and minimum z components are:

$$\Phi_{3/2}^{3/2} = \phi_1^1 \chi_+ \tag{7.129}$$

$$\Phi_{-3/2}^{3/2} = \phi_{-1}^1 \chi_- \tag{7.130}$$

Applying the lowering operator $J_- = L_- + S_-$ to the upper state:

$$L_- \phi_1^1 = \hbar\sqrt{l(l+1) - m(m-1)}\phi_0^1 = \hbar\sqrt{2}\phi_0^1 \tag{7.131}$$

$$S_- \chi_+ = \hbar \chi_- \tag{7.132}$$

so:

$$\Phi_{1/2}^{3/2} = J_- \Phi_{3/2} 3/2 = \hbar \left(\sqrt{2}\phi_0^1 \chi_+ + \phi_1^1 \chi_-\right) \tag{7.133}$$

Normalizing this state:

$$\Phi_{1/2}^{3/2} = \frac{1}{\sqrt{3}} \left(\sqrt{2}\phi_0^1 \chi_+ + \phi_1^1 \chi_-\right) \tag{7.134}$$

We can either lower the state again or we can apply the raising operator to the lowest z component:

$$S_+\phi^1_{-1} = \hbar\sqrt{s(s+1)-m(m+1)}\phi^1_0 = \hbar\sqrt{2}\phi^1_0 \tag{7.135}$$

$$\Phi^{3/2}_{-1/2} = \hbar\left(\sqrt{2}\phi^1_0\chi_- + \phi^1_{-1}\chi_-\right) \tag{7.136}$$

Normalizing this state:

$$\Phi^{3/2}_{-1/2} = \frac{1}{\sqrt{3}}\left(\sqrt{2}\phi^1_0\chi_- + \phi^1_{-1}\chi_+\right) \tag{7.137}$$

Thus, we have obtained the representation of all the states with $j = 3/2$. We can form states orthogonal to the two middle states. For example, write:

$$\Phi^{1/2}_{1/2} = a\psi^1_0\chi_+ + b\phi^1_1\chi_- \tag{7.138}$$

For it to be orthogonal to $\Phi^{3/2}_{1/2}$, we require that:

$$0 = a\sqrt{2} + b \tag{7.139}$$

so that $b = -a\sqrt{2}$. Normalizing the state we obtain:

$$\Phi^{1/2}_{1/2} = \frac{1}{\sqrt{3}}\left(\phi^1_0\chi_+ - \sqrt{2}\phi^1_1\chi_-\right) \tag{7.140}$$

We can either apply the lowering operator to this, or use the same procedure to make an orthogonal combination of the states used to form $\Psi^{3/2}_{-1/2}$ to obtain:

$$\Phi^{1/2}_{-1/2} = \frac{1}{\sqrt{3}}\left(\phi^1_0\chi_- - \sqrt{2}\phi^1_{-1}\chi_+\right) \tag{7.141}$$

For these two states, one can verify by direct calculation:

$$J^2\Phi^{1/2}_{\pm1/2} = \frac{1}{2}\frac{3}{2}\hbar^2\Phi^{1/2}_{\pm1/2} \tag{7.142}$$

and:

$$J_z\Phi^{1/2}_{\pm1/2} = \pm\frac{1}{2}\hbar\Phi^{1/2}_{\pm1/2} \tag{7.143}$$

so that, indeed, they represent the two states of total angular momentum specified by the quantum number $j = 1/2$.

Summarizing the results of this section:

$$j = 3/2: \quad \begin{cases} \Phi^{3/2}_{3/2} = \phi^1_1 \chi_+ \\ \Phi^{3/2}_{1/2} = \frac{1}{\sqrt{3}} \left(\sqrt{2} \phi^1_0 \chi_+ + \phi^1_1 \chi_- \right) \\ \Phi^{3/2}_{-1/2} = \frac{1}{\sqrt{3}} \left(\sqrt{2} \phi^1_0 \chi_- + \phi^1_{-1} \chi_+ \right) \\ \Phi^{3/2}_{-3/2} = \phi^1_{-1} \chi_- \end{cases} \tag{7.144}$$

$$j = 1/2: \quad \begin{cases} \Phi^{1/2}_{1/2} = \frac{1}{\sqrt{3}} \left(\phi^1_0 \chi_+ - \sqrt{2} \phi^1_1 \chi_- \right) \\ \Phi^{1/2}_{-1/2} = \frac{1}{\sqrt{3}} \left(\phi^1_0 \chi_- - \sqrt{2} \phi^1_{-1} \chi_+ \right) \end{cases} \tag{7.145}$$

7.4.2 Clebsch-Gordon Coefficients

What we have been doing in combining two states of angular momentum into a combined state are special cases of the general problem:

$$|s, m\rangle = \sum_{m_1 + m_2 = m} C^{s_1 s_2 s}_{m_1 m_2 m} |s_1 \; m_1\rangle |s_2 \; m_2\rangle \tag{7.146}$$

This equation explicitly takes into account the fact that the z components add to give the total z component. The constants are called the Clebsch-Gordon coefficients, and are tabulated in many places. They are chosen so that the all the states on both sides of the equation are normalized.

The tables also work in reverse, so that:

$$|s_1 m_1\rangle |s_2 m_2\rangle = \sum_s C^{s_1 s_2 s}_{m_1 m_2 m} |s \; m\rangle \tag{7.147}$$

7.4.3 Combining more than two angular momenta

Generally, there are more than two angular momenta to add together. For example, the hydrogen atom consists of a proton that has spin 1/2 and an electron with spin 1/2 and orbital angular momentum. Three angular momenta can be combined in more than one way. Each way produces a complete set of eigenfunctions, but not the same set. Generally the physical interactions determine which set is better. For the hydrogen atom, the spin of the proton has much less effect on the electron than does the interaction between the orbital motion and the electron magnetic moment, so the usual procedure here is to first combine the spin and orbital angular momentum of the electron into the total orbital angular momentum of the electron, and then combine this with the spin angular momentum of the proton.