PHYSICS 355 - Sample Final Test

Name: ______________________________ Soc. Sec. No. ______________

Pledge: ____

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1. Prove that the eigenvalues of a Hermitian transformation are real and that the eigenvectors corresponding to distinct eigenvalues are orthogonal.

2. (a) Using the representation of momentum in coordinate space, show that $[x_i, p_j] = i\hbar\delta_{ij}$, where x_i and p_i are the Cartesian components of position and momentum. Also show that $[x_i, x_j] = [p_i, p_j] = 0$.

(b) Using $\vec{L} = \vec{r} \times \vec{p}$, and the relations of part (a), obtain $[L_x, L_y]$.

3. Find the transmission coefficient for the double delta-function potential:

$$V(x) = -\alpha[\delta(x+a) + \delta(x-a)]$$

where α and a are real constants.

4. (a) Find the expectation value of the potential energy for the ground state and for the $n = 3$, $l = 2$ state of hydrogen. Calculate the expectation value of the kinetic energy for each state.

(b) What is $\langle z^2 \rangle / \langle r^2 \rangle$ for the $n = 3,\ l = 0$ state?

5. Consider the following two 3x3 matrices.

$$A = \begin{pmatrix} 1 & 2 & 2i \\ -1 & 3 & -i \\ -2i & -4i & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix}$$

(a) Which, if either, of the matrices is hermitian?

(b) What is the trace of each matrix?

(c) What is $A + B$?

(d) Calculate the eigenvalues for B.

(e) Calculate one of the eigenvectors for B.

6. (a) What is $L_+ Y_l^l$? (No calculation is needed)

(b) Use the result of part (a) and the fact that $L_z Y_l^l = \hbar l Y_l^l$ to determine Y_l^l to within a normalization constant.

You may want to use the following representations:

$$L_+ = \hbar e^{i\phi} \left(\frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \phi} \right)$$

$$L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}$$

7. (a) Use the methods of linear algebra to obtain the coefficients (to within a phase factor) for the harmonic oscillator ladder operators in the equations

$$a_+ |n\rangle = A_n |n+1\rangle$$

$$a_- |n+1\rangle = B_{n+1} |n\rangle$$

It is useful to note that

$$H = a_+ a_- + \frac{1}{2}\hbar\omega = a_- a_+ - \frac{1}{2}\hbar\omega$$

(b) Given that the ground state of the one-dimensional harmonic oscillator is

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

use the results of part a and explicit differential representation of the ladder operators to derive the normalized wave function ψ_1. (Do not normalize the answer by integrating $|\psi_1|^2$.
The differential representations of the ladder operators are

$$a_+ = \frac{1}{\sqrt{2m}} \left(\frac{\hbar}{i} \frac{d}{dx} + im\omega x \right)$$

$$a_- = \frac{1}{\sqrt{2m}} \left(\frac{\hbar}{i} \frac{d}{dx} - im\omega x \right)$$

Possibly Useful Formulas

$$\int_0^\infty x^n\ e^{-x}\ dx = n!$$

$$\int_{-\infty}^\infty e^{-x^2}\ dx = \sqrt{\pi}$$

$$\int_0^\infty xe^{-x^2}\ dx = \frac{1}{2}$$

$$\int_{-\infty}^\infty x^2e^{-x^2}\ dx = \frac{\sqrt{\pi}}{2}$$

$$e^{ix} = \cos x + i \sin x$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}, \qquad \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\int_{-\infty}^{\infty} e^{-x^2} \; dx = \sqrt{\pi}$$

$$\int_{0}^{\infty} x e^{-x^2} \; dx = \frac{1}{2}$$

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} \; dx = \frac{\sqrt{\pi}}{2}$$

$$\int_{0}^{\infty} x^n e^{-x} \; dx = n!$$

$$\int x^n \, e^{-x^2} \, dx = -\frac{1}{2} x^{n-1} \, e^{-x^2} + \frac{n-1}{2} \int x^{n-2} e^{-x^2} \, dx$$

$$\int x^n \; \sin(ax) \; dx = -\frac{x^n}{a} \cos(ax) + \frac{n}{a} \int x^{n-1} \cos(ax) \, dx$$

$$\int x^n \cos(ax) \; dx = \frac{x^n}{a} \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax) \, dx$$

$$\int x^n \, e^{\alpha x} \, dx = \frac{x^n}{\alpha} e^{\alpha x} - \frac{n}{\alpha} \int x^{n-1} \, e^{\alpha x} \, dx$$

$$\int x^n \, e^{-x} \, dx = -e^{-x} \sum_{m=0}^{n} \left(\frac{n!}{m!} \right) x^m$$

$$\int x^n \, e^{x} \, dx = e^{x} \sum_{m=0}^{n} \left((-1)^{m+n} \frac{n!}{m!} \right) x^m$$