PHYS 355, PS #4 Solutions.

#1

(a) The ground state energy of a square well of width L is,

$$E_0 = \frac{\hbar^2 \pi^2}{2mL^2}$$

using $m_n = 1.67 \times 10^{-27}$ kg,

$$E_0 = 2.05 \text{ MeV}$$

(b) In this case the electron is highly relativistic, so we can't use the above equation obtained by taking $KE = \frac{p^2}{2m}$,

The wave number for the ground state in a I.S.W

$$k = \frac{\pi}{L}$$

using the relativistic eqn for kinetic energy,

$$E_0' = \sqrt{p^2c^2 + m^2c^4} - mc^2$$

$$\simeq pc = \hbar kc$$

$$= 62 \text{ MeV}$$

(ii) Electric potential energy for the electron (for $z = 50$)

$$V = -k_e \cdot \frac{50e^2}{r} \sim -7.2 \text{ MeV}$$

Since $V << E_0'$, the electric potential energy is not sufficient to bind an electron in a nuclear volume.

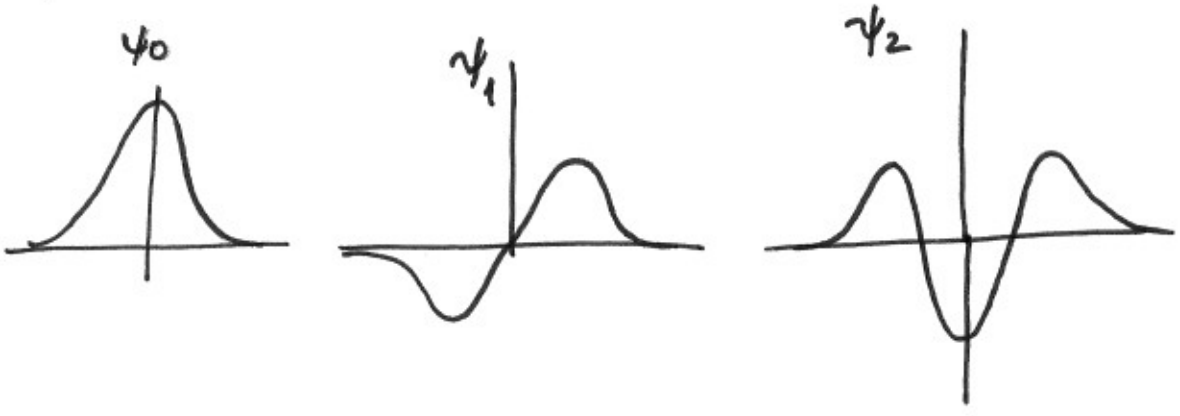
(#2) 2.10

(a) operate twise with a_+ on ψ_0

$$\psi_1 = a_+ \psi_0 = \frac{1}{\sqrt{2\hbar\omega m}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} 2m\omega x e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\psi_2 = \frac{1}{\sqrt{2}} a_+ (a_+ \psi_0) = \frac{1}{\sqrt{2}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(\frac{2m\omega}{\hbar}x^2 - 1\right) e^{\frac{-m\omega}{2\hbar}x^2}$$

(b)



(c) Since ψ_0 and ψ_2 are even whereas ψ_1 is odd $\int \psi_0^* \psi_1 dx$ and $\int \psi_2^* \psi_1 dx$ vanish. we only need to check $\int \psi_2^* \psi_0 dx$

$$\int \psi_2^* \psi_0 dx = \frac{1}{\sqrt{2}} \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} \left(\frac{2m\omega}{\hbar} x^2 - 1\right) e^{-\frac{m\omega}{\hbar}x^2} dx$$

$$= -\sqrt{\frac{m\omega}{2\pi\hbar}} \left(\int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2} dx - \frac{2m\omega}{\hbar} \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega x^2}{\hbar}} dx \right)$$

$$= -\sqrt{\frac{m\omega}{2\pi\hbar}} \left(\sqrt{\frac{\pi\hbar}{m\omega}} - \frac{2m\omega}{\hbar} \frac{\hbar}{2m\omega} \sqrt{\frac{\pi\hbar}{m\omega}} \right) = 0$$

(#3) 2.13

$$\psi(x,0) = A[3\psi_0 + 4\psi_1]$$

(a) $$1 = \int |\psi(x,0)|^2 dx$$

$$= A^2 \int (9|\psi_0|^2 + 12\psi_0^*\psi_1 + 12\psi_1^*\psi_0 + 16|\psi_1|^2) dx$$

$|\psi_0|^2 \Rightarrow 1$, $\psi_0^*\psi_1 \Rightarrow 0$, $\psi_1^*\psi_0 \Rightarrow 0$, $|\psi_1|^2 \Rightarrow 1$

$$1 = 25A^2$$

$$A = 1/5$$

(b) $$\psi_{(x,t)} = \frac{1}{5}\left[3\psi_0 e^{-\frac{iE_0 t}{\hbar}} + 4\psi_1 e^{-\frac{iE_1 t}{\hbar}}\right]$$

$$E_0 = \frac{\hbar\omega}{2} \qquad E_1 = \frac{3\hbar\omega}{2}$$

$$\psi_{(x,t)} = \frac{1}{5}\left[3\psi_0 e^{-\frac{i\omega t}{2}} + 4\psi_1 e^{-\frac{3i\omega t}{2}}\right]$$

$$|\psi_{(x,t)}|^2 = \frac{1}{25}\left[9\psi_0^2 + 12\psi_0\psi_1 e^{-i\omega t} + 12\psi_0\psi_1 e^{i\omega t} + 16\psi_1^2\right]$$

$$= \frac{1}{25}\left[9\psi_0^2 + 16\psi_1^2 + 24\psi_0\psi_1\cos(\omega t)\right]$$

(c)
$$\langle x\rangle = \frac{1}{25}\left[9\int x\psi_0^2\,dx + 16\int x\psi_1^2\,dx + 24\cos\omega t\int x\psi_1\psi_0\,dx\right]$$

but $\int x\psi_0^2\,dx = \int x^2\psi_1^2\,dx = 0$ from symmetry.

$$\int x\psi_0\psi_1\,dx = \sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{2m\omega}{\hbar}}\int x\, e^{-\frac{m\omega}{2\hbar}x^2}\, x\, e^{-\frac{m\omega}{2\hbar}x^2}\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{m\omega}{\hbar}\right)\int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2}\,dx$$

$$= \sqrt{\frac{\hbar}{2m\omega}}$$

$$\therefore \quad \langle x\rangle = \frac{24}{25}\cdot\sqrt{\frac{\hbar}{2m\omega}}\cdot\cos\omega t$$

$$\langle p \rangle = \frac{m\, d\langle x \rangle}{dt} = -\frac{24}{25}\cdot\sqrt{\frac{m\omega\hbar}{2}}\,\sin\omega t$$

with ψ_2 in place of ψ_1 the frequency would be $\dfrac{E_2 - E_0}{\hbar} = 2\omega$

Ehrenfest's theorem says

$$\frac{d\langle p \rangle}{dt} = -\left\langle \frac{\partial V}{\partial x} \right\rangle$$

Here

$$\frac{d\langle p \rangle}{dt} = -\frac{24\omega}{25}\sqrt{\frac{m\omega\hbar}{2}}\,\cos\omega t$$

$$\frac{dV}{dx} = \frac{d}{dx}\left(\tfrac{1}{2}m\omega^2 x^2\right) = m\omega^2 x$$

$$-\left\langle \frac{dV}{dx} \right\rangle = -m\omega^2\langle x \rangle = -m\omega^2\sqrt{\frac{\hbar}{2m\omega}}\cdot\cos\omega t$$

$$= -\frac{24}{25}\cdot\sqrt{\frac{\hbar\omega m}{2}}\cdot\cos\omega t \quad \checkmark$$

(d) you can get $E_0 = \dfrac{\hbar\omega}{2}$ with probability $|C_0|^2 = \dfrac{9}{25}$

or $E_1 = \dfrac{3\hbar\omega}{2}$ with probability $|C_1|^2 = \dfrac{16}{15}$

(#4) 2.14: The new allowed energies are

$$E_n' = \left(n+\tfrac{1}{2}\right)\hbar\omega' \qquad \omega' = 2\omega$$

$$= \hbar\omega,\ 3\hbar\omega,\ \ldots.$$

$\therefore$ Prob. of getting $\frac{\hbar\omega}{2}$ is 0.

Prob. of getting $\hbar\omega$ (the new ground state energy) is $|C_0|^2$

where $C_0 = \int \psi_{(x,0)}\, \psi_0'\, dx$

Colin $\psi_{(x,0)} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$

$$\psi_0' = \left(\frac{m2\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m2\omega}{2\hbar}x^2}$$

$$C_0 = 2^{1/4}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int e^{-\frac{3m\omega}{2\hbar}x^2}\, dx$$

$$= 2^{1/4}\sqrt{\frac{2}{3}}$$

$\therefore$ $P_0 = \frac{2}{3}\sqrt{2} = \underline{0.943}$

(5) 2.15

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2}$$

$$\therefore \quad P = 2\sqrt{\frac{m\omega}{\pi\hbar}} \int_{x_0}^{\infty} e^{-\xi^2} dx$$

$$= 2\sqrt{\frac{m\omega}{\pi\hbar}} \sqrt{\frac{\hbar}{m\omega}} \int_{\xi_0}^{\infty} e^{-\xi^2} d\xi$$

$$\text{but} \quad x_0 = \sqrt{\frac{\hbar}{m\omega}} \Rightarrow \xi_0 = 1$$

$$\therefore \quad P = \frac{2}{\sqrt{\pi}} \cdot \int_1^{\infty} e^{-\xi^2} d\xi$$

$$= 0.157$$

(#6) Half Harmonic oscillator.

The solutions we obtained for S.H.O will work for this system, but with proper boundry conditions at $x=0$

$$\psi_{(x=0)} = 0$$

∴ Harmonic oscillator states with a node at $x=0$;

⇒ $n = 1, 3, 5 \ldots$ states

Energies:

$$E = \frac{3\hbar\omega}{2}, \frac{7\hbar\omega}{2}, \frac{11\hbar\omega}{2} \ldots$$