PHYS 355 PS2 Solutions

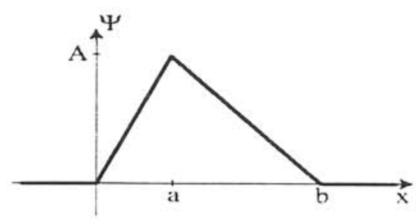
Problem 1.4

(a)

$$1 = \frac{|A|^2}{a^2}\int_0^a x^2 dx + \frac{|A|^2}{(b-a)^2}\int_a^b (b-x)^2 dx = |A|^2 \left\{ \frac{1}{a^2}\left(\frac{x^3}{3}\right)\Big|_0^a + \frac{1}{(b-a)^2}\left(-\frac{(b-x)^3}{3}\right)\Big|_a^b \right\}$$

$$= |A|^2\left[\frac{a}{3} + \frac{b-a}{3}\right] = |A|^2\frac{b}{3} \Rightarrow \boxed{A = \sqrt{\frac{3}{b}}.}$$

(b)



(c) At $\boxed{x = a.}$

(d)

$$P = \int_0^a |\Psi|^2 dx = \frac{|A|^2}{a^2}\int_0^a x^2 dx = |A|^2\frac{a}{3} = \boxed{\frac{a}{b}.} \begin{cases} P = 1 & \text{if } b = a. \checkmark \\ P = 1/2 & \text{if } b = 2a. \checkmark \end{cases}$$

(e)

$$\langle x \rangle = \int x|\Psi|^2 dx = |A|^2\left\{\frac{1}{a^2}\int_0^a x^3 dx + \frac{1}{(b-a)^2}\int_a^b x(b-x)^2 dx\right\}$$

$$= \frac{3}{b}\left\{\frac{1}{a^2}\left(\frac{x^4}{4}\right)\Big|_0^a + \frac{1}{(b-a)^2}\left(b^2\frac{x^2}{2} - 2b\frac{x^3}{3} + \frac{x^4}{4}\right)\Big|_a^b\right\}$$

$$= \frac{3}{4b(b-a)^2}\left[a^2(b-a)^2 + 2b^4 - 8b^4/3 + b^4 - 2a^2b^2 + 8a^3b/3 - a^4\right]$$

$$= \frac{3}{4b(b-a)^2}\left(\frac{b^4}{3} - a^2b^2 + \frac{2}{3}a^3b\right) = \frac{1}{4(b-a)^2}(b^3 - 3a^2b + 2a^3) = \boxed{\frac{2a+b}{4}.}$$

Problem 1.5

(a)

$$1 = \int |\Psi|^2 dx = 2|A|^2\int_0^\infty e^{-2\lambda x}dx = 2|A|^2\left(\frac{e^{-2\lambda x}}{-2\lambda}\right)\Big|_0^\infty = \frac{|A|^2}{\lambda}; \quad \boxed{A = \sqrt{\lambda}.}$$

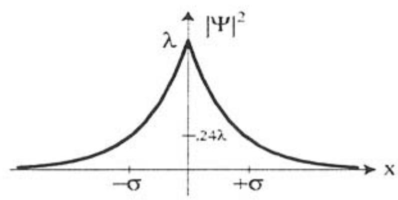
(b)

$$\langle x\rangle = \int x|\Psi|^2 dx = |A|^2 \int_{-\infty}^{\infty} x e^{-2\lambda|x|} dx = \boxed{0.} \qquad \text{[Odd integrand.]}$$

$$\langle x^2\rangle = 2|A|^2 \int_0^{\infty} x^2 e^{-2\lambda x} dx = 2\lambda \left[\frac{2}{(2\lambda)^3}\right] = \boxed{\frac{1}{2\lambda^2}.}$$

(c)

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{1}{2\lambda^2}; \qquad \boxed{\sigma = \frac{1}{\sqrt{2}\lambda}.} \qquad |\Psi(\pm\sigma)|^2 = |A|^2 e^{-2\lambda\sigma} = \lambda e^{-2\lambda/\sqrt{2}\lambda} = \lambda e^{-\sqrt{2}} = 0.2431\lambda.$$



Probability outside:

$$2\int_{\sigma}^{\infty} |\Psi|^2 dx = 2|A|^2 \int_{\sigma}^{\infty} e^{-2\lambda x} dx = 2\lambda \left(\frac{e^{-2\lambda x}}{-2\lambda}\right)\Bigg|_{\sigma}^{\infty} = e^{-2\lambda\sigma} = \boxed{e^{-\sqrt{2}} = 0.2431.}$$

Problem 1.7

$$\langle p \rangle = \int \Psi^* \, (p\Psi) \;\; dx$$

where

$$(p\Psi) = \frac{\hbar}{i}\frac{\partial \Psi}{\partial x}$$

Then

$$\frac{d\langle p \rangle}{dt} = \int \left[\frac{\partial \Psi^*}{\partial t}(p\Psi) + \Psi^*(p\frac{\partial \Psi}{\partial t}) \right] \; dx$$

Using the S.E.

$$\frac{\partial \Psi}{\partial t} = -\frac{i}{\hbar}(p^2/(2m) + V)\Psi$$

$$\frac{\partial \Psi^*}{\partial t} = \frac{i}{\hbar}(p^2/(2m) + V)\Psi^*$$

$$\frac{d\langle p \rangle}{dt} = \frac{i}{\hbar}\int \left[[(\frac{1}{2m}p^2 + V)\Psi^*][p\Psi] - \Psi^*[p(\frac{1}{2m}p^2 + V)\Psi] \right] \; dx$$

$$\frac{d\langle p \rangle}{dt} = \frac{i}{2m\hbar}\int \left[[p^2\Psi^*][p\Psi] - \Psi^*[p^3\Psi] \right] \; dx + \frac{i}{\hbar}\int [(V\Psi^*)(p\Psi) - \Psi^*(pV\Psi)] \; dx$$

Integrating the second term in the first integral by parts, and using the fact that the wave function vanishes at infinity,

$$\frac{d\langle p \rangle}{dt} = \frac{i}{2m\hbar}\int \left[(p^2\Psi^*)(p\Psi) + p\Psi^*(p^2\Psi) \right] \; dx + \frac{i}{\hbar}\int [(V\Psi^*)(p\Psi) - \Psi^*(pV\Psi)] \; dx$$

The first integrand $(p^2\Psi^*)(p\Psi) + p\Psi^*(p^2\Psi)$ can be written as $p\,[(p\Psi^*)(p\Psi)]$, which makes it a perfect differntial that integrates to 0. We are then left with

$$\frac{d\langle p \rangle}{dt} = +\frac{i}{\hbar}\int [(V\Psi^*)(p\Psi) - \Psi^*(pV\Psi)] \; dx$$

Using the chain rule for the operation of p on $V\Psi$,

$$pV\Psi = (pV)\Psi + V(p\Psi) = \frac{\hbar}{i}\frac{dV}{dx}\Psi + Vp\psi$$

Since $V(x)$ is a just a function of x, its order in the integrand doesn't matter. Therefore,

$$\frac{d\langle p \rangle}{dt} = -\int \Psi^* \frac{dV}{dx}\Psi \; dx = \langle -\frac{dV}{dx} \rangle$$

Problem 1.8

Suppose Ψ satisfies the Schrödinger equation *without* V_0: $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$. We want to find the solution Ψ_0 *with* V_0: $i\hbar\frac{\partial\Psi_0}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi_0}{\partial x^2} + (V+V_0)\Psi_0$.

Claim: $\Psi_0 = \Psi e^{-iV_0t/\hbar}$.

Proof: $i\hbar\frac{\partial\Psi_0}{\partial t} = i\hbar\frac{\partial\Psi}{\partial t}e^{-iV_0t/\hbar} + i\hbar\Psi\left(-\frac{iV_0}{\hbar}\right)e^{-iV_0t/\hbar} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\right]e^{-iV_0t/\hbar} + V_0\Psi e^{-iV_0t/\hbar}$

$= -\frac{\hbar^2}{2m}\frac{\partial^2\Psi_0}{\partial x^2} + (V+V_0)\Psi_0.$ QED

This has *no* effect on the expectation value of a dynamical variable, since the extra phase factor, being independent of x, cancels out in Eq. 1.36.

Problem 1.9

(a)

$$1 = 2|A|^2\int_0^\infty e^{-2amx^2/\hbar}dx = 2|A|^2\frac{1}{2}\sqrt{\frac{\pi}{(2am/\hbar)}} = |A|^2\sqrt{\frac{\pi\hbar}{2am}}; \quad \boxed{A = \left(\frac{2am}{\pi\hbar}\right)^{1/4}.}$$

(b)

$$\frac{\partial\Psi}{\partial t} = -ia\Psi; \quad \frac{\partial\Psi}{\partial x} = -\frac{2amx}{\hbar}\Psi; \quad \frac{\partial^2\Psi}{\partial x^2} = -\frac{2am}{\hbar}\left(\Psi + x\frac{\partial\Psi}{\partial x}\right) = -\frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\Psi.$$

Plug these into the Schrödinger equation, $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$:

$$V\Psi = i\hbar(-ia)\Psi + \frac{\hbar^2}{2m}\left(-\frac{2am}{\hbar}\right)\left(1 - \frac{2amx^2}{\hbar}\right)\Psi$$

$$= \left[\hbar a - \hbar a\left(1 - \frac{2amx^2}{\hbar}\right)\right]\Psi = 2a^2mx^2\Psi, \quad \text{so} \quad \boxed{V(x) = 2ma^2x^2.}$$

(c)

$$\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi|^2dx = \boxed{0.} \qquad \text{[Odd integrand.]}$$

$$\langle x^2\rangle = 2|A|^2\int_0^\infty x^2e^{-2amx^2/\hbar}dx = 2|A|^2\frac{1}{2^2(2am/\hbar)}\sqrt{\frac{\pi\hbar}{2am}} = \boxed{\frac{\hbar}{4am}.}$$

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{0.}$$

$$\langle p^2\rangle = \int\Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\Psi dx = -\hbar^2\int\Psi^*\frac{\partial^2\Psi}{\partial x^2}dx$$

$$= -\hbar^2\int\Psi^*\left[-\frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\Psi\right]dx = 2am\hbar\left\{\int|\Psi|^2dx - \frac{2am}{\hbar}\int x^2|\Psi|^2dx\right\}$$

$$= 2am\hbar\left(1 - \frac{2am}{\hbar}\langle x^2\rangle\right) = 2am\hbar\left(1 - \frac{2am}{\hbar}\frac{\hbar}{4am}\right) = 2am\hbar\left(\frac{1}{2}\right) = \boxed{am\hbar.}$$

Problem 1.17

(a)

$$1 = |A|^2 \int_{-a}^{a} \left(a^2 - x^2\right)^2 dx = 2|A|^2 \int_0^a \left(a^4 - 2a^2x^2 + x^4\right) dx = 2|A|^2 \left[a^4 x - 2a^2\frac{x^3}{3} + \frac{x^5}{5}\right]\Bigg|_0^a$$

$$= 2|A|^2 a^5 \left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{16}{15} a^5 |A|^2, \text{ so } \boxed{A = \sqrt{\frac{15}{16a^5}}.}$$

(b)

$$\langle x \rangle = \int_{-a}^{a} x |\Psi|^2 \, dx = \boxed{0.} \quad \text{(Odd integrand.)}$$

(c)

$$\langle p \rangle = \frac{\hbar}{i} A^2 \int_{-a}^{a} \left(a^2 - x^2\right) \underbrace{\frac{d}{dx}\left(a^2 - x^2\right)}_{-2x} dx = \boxed{0.} \quad \text{(Odd integrand.)}$$

Since we only know $\langle x \rangle$ at $t = 0$ we cannot calculate $d\langle x \rangle / dt$ directly.

(d)

$$\begin{aligned}\langle x^2\rangle &= A^2\int_{-a}^{a} x^2\left(a^2-x^2\right)^2 dx = 2A^2\int_0^a \left(a^4x^2-2a^2x^4+x^6\right)dx \\ &= 2\frac{15}{16a^5}\left[a^4\frac{x^3}{3}-2a^2\frac{x^5}{5}+\frac{x^7}{7}\right]\Big|_0^a = \frac{15}{8a^5}\left(a^7\right)\left(\frac{1}{3}-\frac{2}{5}+\frac{1}{7}\right) \\ &= \frac{\cancel{15}a^2}{8}\left(\frac{35-42+15}{\cancel{3}\cdot\cancel{5}\cdot 7}\right) = \frac{a^2}{8}\cdot\frac{8}{7} = \boxed{\frac{a^2}{7}.}\end{aligned}$$

(e)

$$\langle p^2\rangle = -A^2\hbar^2\int_{-a}^{a}\left(a^2-x^2\right)\underbrace{\frac{d^2}{dx^2}\left(a^2-x^2\right)}_{-2}dx = 2A^2\hbar^2 2\int_0^a\left(a^2-x^2\right)dx$$

$$= 4\cdot\frac{15}{16a^5}\hbar^2\left(a^2x-\frac{x^3}{3}\right)\Big|_0^a = \frac{15\hbar^2}{4a^5}\left(a^3-\frac{a^3}{3}\right) = \frac{15\hbar^2}{4a^2}\cdot\frac{2}{3} = \boxed{\frac{5}{2}\frac{\hbar^2}{a^2}.}$$

(f)

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{1}{7}a^2} = \boxed{\frac{a}{\sqrt{7}}.}$$

(g)

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{5}{2}\frac{\hbar^2}{a^2}} = \boxed{\sqrt{\frac{5}{2}}\frac{\hbar}{a}.}$$

(h)

$$\sigma_x\sigma_p = \frac{a}{\sqrt{7}}\cdot\sqrt{\frac{5}{2}}\frac{\hbar}{a} = \sqrt{\frac{5}{14}}\hbar = \sqrt{\frac{10}{7}}\frac{\hbar}{2} > \frac{\hbar}{2}.\ \checkmark$$
