1. a) Need to keep exchange symmetry, so
(4 pts)

$$\Psi_A = \frac{1}{\sqrt{N}}\Big[\psi_a(1)\,\psi_b(2)\,\psi_b(3)\ldots\ \psi_b(N)$$
$$+\,\psi_b(1)\,\psi_a(2)\,\psi_b(3)\ \ldots\ \psi_b(N)$$
$$+\ \ldots$$
$$+\,\psi_b(1)\,\psi_b(2)\ \ldots\ \ \psi_b(N-1)\,\psi_a(N)\Big]$$

b) Here $P_{a\to b} = \frac{|V_{ab}|^2}{\hbar^2}\,\frac{\sin^2\frac{\Delta t}{2}}{\Delta^2}$ for one particle.
(6 pts)

For N particles, $P_{A\to B} = \frac{|V_{AB}|^2}{\hbar^2}\,\frac{\sin^2\frac{\Delta t}{2}}{\Delta^2}$

where $V_{AB} = \langle \Psi_A | V | \Psi_B \rangle$

$$= \frac{1}{\sqrt{N}}\Big[\psi_a(1)\,\psi_b(2)\ \ldots\ \psi_b(N) + \ldots\Big]\Big[V(1)+V(2)+\ldots+V(N)\Big]$$

$$\times\ \psi_b(1)\,\psi_b(2)\ldots\ \psi_b(N)$$

Have $\langle \psi_b(n)|V(n)|\psi_b(n)\rangle = 0$, so for each V, only one term survives

$$V_{AB} = \frac{1}{\sqrt{N}}\Big[\langle\psi_a(1)|V(1)|\psi_b(1)\rangle + \langle\psi_a(2)|V(2)|\psi_b(2)\rangle + \ldots \langle\psi_a(N)|V(N)|\psi_b(N)\rangle\Big]$$

But each of these is the same, $= V_{ab}$

$$V_{AB} = \frac{N}{\sqrt{N}}\ V_{ab} = \sqrt{N}\ V_{ab}$$

So $P_{A\to B} = N\,\frac{|V_{ab}|^2}{\hbar^2}\,\frac{\sin^2\frac{\Delta t}{2}}{\Delta^2} = N\,P_{a\to b}$

Factor of N larger.

2. Need to normalize ψ:

$$\int_{-\infty}^{\infty} |\psi|^2 dx = 2A^2 \int_0^{\infty} e^{-2x/a} dx = 2A^2 \cdot \frac{a}{2} = 1$$

$$A = \frac{1}{\sqrt{a}} \quad (2\text{ pts})$$

Then get $\langle H \rangle = \langle -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) \rangle$

Kinetic energy

$$-\frac{\hbar^2}{2m}\frac{1}{a}\int_{-\infty}^{\infty} e^{-|x|/a}\frac{d^2}{dx^2} e^{-|x|/a} dx$$

except at $x=0$, have $\frac{d^2}{dx^2} e^{-|x|/a} = \frac{1}{a^2} e^{-|x|/a}$

But at $x=0$, have discontinuity in slope

$$\frac{d}{dx} e^{-|x|/a} = \begin{cases} \frac{1}{a} & (x<0) \\ -\frac{1}{a} & (x>0) \end{cases}$$

step fcn. Derivative of step is δ-fcn

$$\frac{d^2}{dx^2} e^{-|x|/a} = -\frac{2}{a}\delta(x) \quad \text{for } x=0$$

So $$KE = -\frac{\hbar^2}{2m}\frac{1}{a}\left[\int_{-\infty}^{\infty}\frac{1}{a^2} e^{-2|x|/a} dx - \frac{2}{a}\int_{-\infty}^{\infty} e^{-2|x|/a}\delta(x)\, dx\right]$$

$$= -\frac{\hbar^2}{2ma}\left[\frac{2}{a^2}\int_0^{\infty} e^{-2x/a} dx - \frac{2}{a}\right]$$

$$\left[\frac{2}{a^2}\cdot\left(\frac{a}{2}\right) - \frac{2}{a}\right]$$

$$\left[-\frac{1}{a}\right]$$

$$KE = \frac{\hbar^2}{2ma^2} \quad (3\text{ pts})$$

Potential:

$$\langle V \rangle = \frac{1}{a}\int_{-\infty}^{\infty} e^{-|x|/a} \, \alpha |x| e^{-|x|/a} \, dx$$

$$= \frac{2\alpha}{a}\int_{0}^{\infty} x e^{-2x/a} \, dx$$

$$= \frac{2\alpha}{a}\left(\frac{a}{2}\right)^2 = \frac{\alpha a}{2} \qquad (3\text{ pts})$$

$$\langle H \rangle = \frac{\hbar^2}{2ma^2} + \frac{\alpha a}{2}$$

$$\frac{\partial \langle H \rangle}{\partial a} = -\frac{\hbar^2}{ma^3} + \frac{\alpha}{2} = 0$$

$$\frac{1}{a^3} = \frac{m\alpha}{2\hbar^2}$$

$$a = \left(\frac{2\hbar^2}{m\alpha}\right)^{1/3}$$

$$\langle H \rangle = \frac{\hbar^2}{4m}\left(\frac{m\alpha}{2\hbar^2}\right)^{2/3} + \frac{\beta}{4}\left(\frac{2\hbar^2}{m\alpha}\right)^{1/3}$$

$$= \frac{1}{4}\left(\frac{\hbar^2\alpha^2}{m}\right)^{1/3}\left(\frac{1}{2^{2/3}} + 2^{1/3}\right)$$

$$= \frac{1}{4}\left(\frac{2\hbar^2\alpha^2}{m}\right)^{1/3}\left(\frac{1}{2} + 1\right)$$

$$\boxed{\langle H \rangle = \frac{3}{8}\left(\frac{2\hbar^2\alpha^2}{m}\right)^{1/3}} = 0.47\left(\frac{\hbar^2\alpha^2}{m}\right)^{1/3}$$

(2 pts)

Gaussian gives $\frac{3}{2}\left(\frac{\alpha^2\hbar^2}{2\pi m}\right)^{1/3} = 0.81\left(\frac{\hbar^2\alpha^2}{m}\right)^{1/3}$

3. Born approximation says

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{i(\vec{k}'-\vec{k})\cdot\vec{r}_0}\, V(\vec{r}_0)\, d^3r_0$$

$$\vec{k}' = k\hat{z}$$

$$\vec{k} = k\hat{r} \quad \text{for scattering in direction } \hat{r}$$

Here $k = \frac{\sqrt{2mE}}{\hbar}$

Integral is easy. Define $\vec{q} = \vec{k}' - \vec{k}$

$$f = -\frac{mV_0}{2\pi\hbar^2}\int_{-a}^{a} e^{iq_x x}\,dx \int_{-a}^{a} e^{iq_y y}\,dy \int_{-a}^{a} e^{iq_z z}\,dz$$

$$= -\frac{mV_0}{2\pi\hbar^2}\left[\frac{e^{iq_x a}-e^{-iq_x a}}{iq_x}\right]\left[\frac{e^{iq_y a}-e^{-iq_y a}}{iq_y}\right] \times\left[\frac{e^{iq_z a}-e^{-iq_z a}}{iq_z}\right]$$

$$\boxed{f = -\frac{4mV_0}{\pi\hbar^2}\,\frac{\sin q_x a}{q_x}\cdot\frac{\sin q_y a}{q_y}\cdot\frac{\sin q_z a}{q_z}}$$

Since $q = \vec{k}' - \vec{k}$,

$$q_x = -k_x = -k\sin\theta\cos\phi$$

$$q_y = -k_y = -k\sin\theta\sin\phi$$

$$q_z = k - k_z = k(1-\cos\theta)$$

Using normal spherical coordinates.

4. Use regular perturbation theory. Don't couple to spin, so no degeneracies. $H' = -eV = -e\mathcal{E} r\cos\Theta$

First order shift

$$E^{(1)} = \langle \psi^{0}_{100} | H' | \psi^{0}_{100} \rangle = -e\mathcal{E} \langle \psi_{100} | z | \psi_{100} \rangle$$

$= 0$ since s-state is spherically symmetric

Second order shift

$$E^{(2)}_{n} = \sum_{m\neq n} \frac{|\langle \psi^{0}_{m} | H' | \psi^{0}_{n} \rangle|^2}{E^{(0)}_{n} - E^{(0)}_{m}}$$

2 pts: set up
3 pts: find nonzero terms
5 pts: integrals

we only need to consider states $\psi_{n\ell m}$ with $n \leq 3$

Need $\langle \psi_{n\ell m} | z | \psi_{100} \rangle$ for these states.

Note: if $\ell = 0$, matrix element $= 0$ due to spherical symmetry

If $m \neq 0$, $\psi_{n\ell m}$ has term $e^{im\phi}$. This is only ϕ dependence, and it gives zero when integrated.

Leaves only ψ_{210}, ψ_{310} and ψ_{320}.

Do angular and radial parts seperately.

Angular part. $\langle Y^{0}_{1} | \cos\Theta | Y^{0}_{0} \rangle$ $\quad (z = r\cos\Theta)$

and $\langle Y^{0}_{2} | \cos\Theta | Y^{0}_{0} \rangle$

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

Note that $\cos\theta \, Y_0^0$ is proportional to Y_1^0

So $\langle Y_2^0 | \cos\theta | Y_0^0 \rangle \propto \langle Y_2^0 | Y_1^0 \rangle$

But Y_ℓ^m's are orthogonal, so this is zero

Need $\langle Y_1^0 | \cos\theta | Y_0^0 \rangle$

$$= \frac{\sqrt{3}}{4\pi} \int_0^\pi \cos^2\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{\sqrt{3}}{2} \left(-\frac{\cos^3\theta}{3} \right) \Big|_0^\pi$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{2}{3} = \boxed{\frac{1}{\sqrt{3}}}$$

Radial part:

Need $\langle R_{21} | r | R_{10} \rangle$ and $\langle R_{31} | r | R_{10} \rangle$

$$R_{10} = \frac{2}{a^{3/2}} e^{-r/a} \qquad R_{21} = \frac{1}{\sqrt{24}} \frac{1}{a^{3/2}} \frac{r}{a} e^{-r/2a}$$

$$R_{31} = \frac{8}{27\sqrt{6}} \frac{1}{a^{3/2}} \left(1 - \frac{r}{6a}\right) \frac{r}{a} e^{-r/3a}$$

$$\langle R_{21} | r | R_{10} \rangle = \frac{1}{\sqrt{6}} \frac{1}{a^4} \int_0^\infty \left(r e^{-r/2a}\right) r \left(e^{-r/a}\right) r^2 \, dr$$

$$= \frac{1}{\sqrt{6}} \frac{1}{a^4} \int_0^\infty r^4 e^{-3r/2a} \, dr$$

$$= \frac{1}{\sqrt{6}} \frac{1}{a^4} \cdot 4! \left(\frac{2a}{3}\right)^5$$

$$= \frac{24}{\sqrt{6}} a \cdot \frac{2^5}{3^5} = \frac{1}{\sqrt{6}} a \frac{2^8}{3^4} = \boxed{\frac{2^{15/2}}{3^{9/2}} a}$$

$$\langle R_{31}|r|R_{10}\rangle = \frac{16}{27\sqrt{6}}\,\frac{1}{a^3}\int_0^\infty \left(1-\frac{r}{6a}\right)\frac{r}{a}e^{-r/3a}\cdot r\cdot e^{-r/a}\,r^2\,dr$$

$$= \frac{2^4}{3^3\sqrt{6}}\,\frac{1}{a^4}\left[\int_0^\infty r^4 e^{-4r/3a}\,dr - \frac{1}{6a}\int_0^\infty r^5 e^{-4r/3a}\,dr\right]$$

$$= \frac{2^4}{3^3\sqrt{6}}\,\frac{1}{a^4}\left[4!\left(\frac{3a}{4}\right)^5 - \frac{1}{6a}\cdot 5!\left(\frac{3a}{4}\right)^6\right]$$

$$= \frac{2^4}{3^3\sqrt{6}}\,a\,\underbrace{4!}_{=24=3\cdot 8}\left(\frac{3}{4}\right)^5\left[1-\frac{1}{6}\cdot 5\cdot\frac{3}{4}\right]$$

$$= \frac{2^4}{3^3\sqrt{6}}\,a\,\frac{3^6}{2^7}\left[\underbrace{1-\frac{5}{8}}_{=3/8}\right]$$

$$= \frac{3^4}{2^6\sqrt{6}}\,a = \boxed{\frac{3^{7/2}}{2^{13/2}}\,a}$$

So $\langle\psi_{210}|z|\psi_{100}\rangle = \dfrac{2^{15/2}}{3^5}\,a = 0.74\,a$

$\langle\psi_{310}|z|\psi_{100}\rangle = \dfrac{3^3}{2^{13/2}}\,a = 0.30\,a$

and $E^{(2)} = e^2\mathcal{E}^2a^2\left[\dfrac{2^{15}}{3^{10}}\,\dfrac{1}{E_1-E_2} + \dfrac{3^6}{2^{13}}\,\dfrac{1}{E_1-E_3}\right]$

$$E_n = \frac{E_1}{n^2}$$

$$= \frac{e^2\mathcal{E}^2a^2}{E_1}\left[\frac{2^{15}}{3^{10}}\,\underbrace{\frac{1}{1-\frac{1}{4}}}_{\frac{4}{3}} + \frac{3^6}{2^{13}}\,\underbrace{\frac{1}{1-\frac{1}{9}}}_{\frac{9}{8}}\right]$$

$$= \frac{e^2\mathcal{E}^2a^2}{E_1}\left[\frac{2^{17}}{3^{11}} + \frac{3^8}{2^{15}}\right]$$

$$[0.74 + 0.20]$$

$\boxed{E^{(2)} = 0.94\,\dfrac{e^2\mathcal{E}^2a^2}{E_1}}$ $\qquad E_1 = -13.6\,eV$

5. This is a two level sytem, use time dependent perturbation theory.

Without perturbation, have states $\chi_+ = \psi_a$, $\chi_- = \psi_b$

$$E_a = \mu_B B_0 \qquad E_b = -\mu_B B_0 \qquad \mu_B = \frac{e\hbar}{2m}$$

$$\text{So } \omega_0 = \frac{E_a - E_b}{\hbar} = \frac{2\mu_B B_0}{\hbar}$$

Use 1st order PT, correctly: 6/10

Perturbation is $H' = 2\frac{\mu_B}{\hbar} B_1 S_x$

$$\text{Know } S_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{So } H'_{ab} = \mu_B B_1 = H'_{ba}$$

Have constant perturbation.

Know answer from problem 9.2:

$$C_b(t) = \frac{2H'_{ba}}{i\hbar\omega} \sin\frac{\omega t}{2} \qquad \omega = \sqrt{\omega_0^2 + 4\frac{|H_{ab}'|^2}{\hbar^2}}$$

$$\text{So } P_{a\to b}(t) = |c_b|^2 = \frac{4\mu_B^2 B_1^2}{\sqrt{4\mu_B^2 B_0^2 + 4\mu_B^2 B_1^2}} \sin^2\frac{\omega t}{2}$$

$$= \boxed{\frac{B_1^2}{\sqrt{B_0^2 + B_1^2}} \sin^2 \frac{\mu_B t}{\hbar}\sqrt{B_0^2 + B_1^2}}$$

To derive this from scratch, use

$$\dot{c}_a = -i\Omega e^{-i\omega_0 t} c_b$$

$$\dot{c}_b = -i\Omega e^{i\omega_0 t} c_a \qquad \text{for } \Omega = \frac{H'_{ab}}{\hbar}$$

$$\ddot{c}_b = -i\Omega(i\omega_0 c_a + \dot{c}_a)e^{i\omega_0 t}$$

$$= -i\Omega\left(i\omega_0 \frac{\dot{c}_b}{-i\Omega} e^{-i\omega_0 t} - i\Omega e^{-i\omega_0 t} c_b\right) e^{i\omega_0 t}$$

$$= i\omega_0 \dot{c}_b - \Omega^2 c_b$$

$$\ddot{c}_b - i\omega_0 \dot{c}_b + \Omega^2 c_b = 0$$

Try $c_b = e^{\lambda t}$: $\qquad \lambda^2 - i\omega_0 \lambda + \Omega^2 = 0$

$$\lambda = \frac{1}{2}\left(i\omega_0 \pm \sqrt{-\omega_0^2 - 4\Omega^2}\right)$$

$$= \frac{i}{2}(\omega_0 \pm \omega)$$

$$c_b(t) = Ae^{i\frac{\omega_0+\omega}{2}t} + Be^{i\frac{\omega_0-\omega}{2}t}$$

$$c_b(0) = 0, \quad \text{so} \quad B = -A$$

$$c_b(t) = Ae^{i\frac{\omega_0 t}{2}}\left(e^{i\frac{\omega t}{2}} - e^{-i\frac{\omega t}{2}}\right)$$

$$= 2iAe^{i\omega_0 t/2} \sin\frac{\omega t}{2}$$

$$\dot{c}_b(0) = A\left[i\frac{\omega_0+\omega}{2} - i\frac{\omega_0-\omega}{2}\right] = iA\omega$$

But $\dot{c}_b(0) = -i\Omega c_a(0) = -i\Omega = iA\omega$

So $A = -\frac{\Omega}{\omega}$ $\qquad \boxed{c_b(t) = \frac{2\Omega}{i\omega}\sin\frac{\omega t}{2}}$ ✓

6. By adiabatic theorem, particle remains in the ground state.

Energy for square well size a: $E_1 = \frac{\pi^2\hbar^2}{2ma^2}$

So for size $2a$

$$E_1 = \frac{\pi^2\hbar^2}{2m(2a)^2} = \boxed{\frac{\pi^2\hbar^2}{8ma^2}}$$