Lecture 1 - Identical Particles

Start with Ch 5 - Identical Particles

Begin by consider quantum system of two particles
Particle 1 position $\vec{r}_1$
" 2 " $\vec{r}_2$

Describe with two particle wavefunction
$\psi(\vec{r}_1, \vec{r}_2, t)$

Satisfies Schrodinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi$$

With $H = -\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 + V(\vec{r}_1, \vec{r}_2, t)$

$$\nabla_1^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial z_1^2}, \quad \text{sim for } \nabla_2$$

Normalization

$$\int |\psi|^2 d^3r_1 d^3r_2 = 1$$

Interpret $|\psi(\vec{r}_1, \vec{r}_2)|^2$ = prob density to find
particle 1 at $\vec{r}_1$
particle 2 at $\vec{r}_2$

[In general, have many body wave function
$\psi(\vec{r}_1, \vec{r}_2, \dots \vec{r}_N, t)$ ... same idea]

If H doesn't depend on time, can separate variables:

$$\psi(\vec{r}_1, \vec{r}_2, t) = \psi(\vec{r}_1, \vec{r}_2)\, e^{-iEt/\hbar}$$

with $-\frac{\hbar^2}{2m}\nabla_1^2\psi - \frac{\hbar^2}{2m}\nabla_2^2\psi + V\psi = E\psi$

* very hard to solve!

- If $V(\vec{r}_1,\vec{r}_2) = V(\vec{r}_1) + V(\vec{r}_2)$
  particles are independent

  Can separate further: $\psi(\vec{r}_1,\vec{r}_2) = \psi_1(\vec{r}_1)\,\psi_2(\vec{r}_2)$

  with $-\frac{\hbar^2}{2m}\nabla_1^2\psi_1 + V(\vec{r}_1)\psi_1 = E_1\psi_1$

  $-\frac{\hbar^2}{2m}\nabla_2^2\psi_2 + V(\vec{r}_2)\psi_2 = E_2\psi_2$

  $E = E_1 + E_2$

  Just two copies of single-particle problem

- More interesting when particles interact:
  Typically $V(\vec{r}_1,\vec{r}_2) = V(\vec{r}_1 - \vec{r}_2)$
  (no external potential)

  Then can't separate r_1 + r_2

  Can separate different variables:
  center-of-mass coordinates

[Problem 5.1]

Define $\vec{r} = \vec{r}_1 - \vec{r}_2$ relative coordinate

$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$ COM coordinate

Convert Schr. eqn. to these coordinates

$$\vec{r}_2 = \vec{r}_1 - \vec{r}$$

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2(\vec{r}_1 - \vec{r})}{m_1+m_2} = \vec{r}_1 - \frac{m_2}{m_1+m_2}\vec{r}$$

So $\boxed{\vec{r}_1 = \vec{R} + \frac{m_2}{m_1+m_2}\vec{r}}$

$$\vec{r}_2 = \vec{r}_1 - \vec{r} = \vec{R} + \left(\frac{m_2}{m_1+m_2} - 1\right)\vec{r}$$

$$\boxed{\vec{r}_2 = \vec{R} - \frac{m_1}{m_1+m_2}\vec{r}}$$

Define $M = m_1 + m_2$
= total mass

Need to convert derivatives $\vec{\nabla}_1$ + $\vec{\nabla}_2$

Have $\frac{\partial}{\partial x_1} = \frac{\partial X}{\partial x_1}\frac{\partial}{\partial X} + \frac{\partial x}{\partial x_1}\frac{\partial}{\partial x}$

$$X = \frac{m_1 x_1 + m_2 x_2}{M} \qquad x = x_1 - x_2$$

So $\frac{\partial}{\partial x_1} = \frac{m_1}{M}\frac{\partial}{\partial X} + \frac{\partial}{\partial x}$

Similarly for other coordinates, get

$$\vec{\nabla}_1 = \frac{m_1}{M}\vec{\nabla}_R + \vec{\nabla}_r$$

$$\vec{\nabla}_2 = \frac{m_2}{M}\vec{\nabla}_R - \vec{\nabla}_r$$

So $\nabla_1^2 = \frac{m_1^2}{M^2}\nabla_R^2 + \nabla_r^2 + 2\frac{m_1}{M}\vec{\nabla}_R \cdot \vec{\nabla}_r$

$$\nabla_2^2 = \frac{m_2^2}{M^2}\nabla_R^2 + \nabla_r^2 - 2\frac{m_2}{M}\vec{\nabla}_R \cdot \vec{\nabla}_r$$

Put into Schr. Eqn:

$$\left[-\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(\vec{r})\right]\psi = E\psi$$

$$\left[-\frac{\hbar^2}{2}\frac{m_1}{M^2}\nabla_R^2 - \frac{\hbar^2}{2m_1}\nabla_r^2 - \frac{\hbar^2}{M}\vec{\nabla}_R\cdot\vec{\nabla}_r\right.$$

$$\left. - \frac{\hbar^2}{2}\frac{m_2}{M^2}\nabla_R^2 - \frac{\hbar^2}{2m_2}\nabla_r^2 + \frac{\hbar^2}{M}\vec{\nabla}_R\cdot\vec{\nabla}_r\right] = (E-V)\psi$$

or

$$\left[-\frac{\hbar^2}{2M}\nabla_R^2 - \frac{\hbar^2}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\nabla_r^2 + V(\vec{r})\right]\psi = E\psi$$

Define $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$

$\mu \equiv$ reduced mass

Then separate:

$$-\frac{\hbar^2}{2M}\nabla_R^2\psi = E_R\psi \quad \rightarrow \text{free particle equation}$$

$$-\frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\vec{r})\psi = E_r\psi \quad \rightarrow \text{single particle in potential}$$

Total $E = E_r + E_R$

Usually don't even think about $\vec{R}$

Know solutions are plane waves $\sim e^{i\vec{k}\cdot\vec{r}}$

You already know how to solve all the single particle problems that have exact solutions.

So you can apply that directly to two-body systems

Note: doesn't work if there's an external potential $V(\vec{R})$!

Note, in general, particles will also have spin

$$\psi(\vec{r}_1, \vec{r}_2, t) \rightarrow \psi(\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2, t)$$

Often (not always) the case that spatial & spin parts separate

$$\rightarrow \psi(\vec{r}_1, \vec{r}_2, t)\, \chi(\vec{s}_1, \vec{s}_2, t)$$

Then above discussion applies to spatial part

Use techniques from Ch 4 to handle spin part
(Know how to combine two spins)

Exchange Symmetry

Now suppose our particles are the same type
two electrons, for instance.

Then particles are indistinguishable: can't tell one from the other

Must have $H(\vec{r}_1, \vec{r}_2) = H(\vec{r}_2, \vec{r}_1)$
symmetric under particle exchange

Use symmetry to define a new operator that commutes with H:

P = exchange operator

$$P\,\psi(\vec{r}_1, \vec{r}_2) = \psi(\vec{r}_2, \vec{r}_1)$$

To see that P commutes with H:

$$PH\psi = P\left[H(\vec{r}_1,\vec{r}_2)\,\psi(\vec{r}_1,\vec{r}_2)\right] = H(\vec{r}_2,\vec{r}_1)\,\psi(\vec{r}_2,\vec{r}_1)$$

$$= H(\vec{r}_1,\vec{r}_2)\,\psi(\vec{r}_2,\vec{r}_1)$$

$$= HP\psi$$

So $[P,H]=0$

So we can find eigenstates of H that are also eigenstates of P

What are eigenvalues of P?

Have $P^2\psi(\vec{r}_1,\vec{r}_2) = P\psi(\vec{r}_2,\vec{r}_1) = \psi(\vec{r}_1,\vec{r}_2)$

So $P^2 = 1$

Suppose ϕ is an eigenstate of P

Then $P\phi = \lambda\phi$

$P^2\phi = \lambda^2\phi = \phi$

So $\lambda^2 = 1 \Rightarrow \lambda = \pm 1$

Two eigenstates: $\lambda = +1 \Rightarrow$ even exchange symmetry

$\lambda = -1 \Rightarrow$ odd exchange symmetry

Eigenstates have form

$$\psi(\vec{r}_1,\vec{r}_2) = A\left[\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) \pm \psi_a(\vec{r}_2)\psi_b(\vec{r}_1)\right]$$

$+$ for even, $-$ for odd