Lecture 4 - Identical Particles

Start with Ch 5 - Identical Particles

Begin by consider quantum system of two particles

Particle 1 position $\vec{r}_1$

Particle 2 position $\vec{r}_2$

Describe with two particle wavefunction $\psi(\vec{r}_1, \vec{r}_2, t)$

Satisfies Schrodinger equation

$$i\hbar \frac{\partial}{\partial t} \psi = H \psi$$

with

$$H = -\frac{\hbar^2}{2m} \nabla_1^2 - \frac{\hbar^2}{2m} \nabla_2^2 + U(\vec{r}_1, \vec{r}_2, t)$$

$\nabla_i^2 = \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + \frac{\partial^2}{\partial z_i^2}$, sim for $\nabla_2$

Normalization

$$\int (\psi)^2 d^3r_1 d^3r_2 = 1$$

Interpret $|\psi(\vec{r}_1, \vec{r}_2)|^2 = \text{prob density to find particle 1 at } \vec{r}_1$

particle 2 at $\vec{r}_2$

[In general, have many body wave function $\psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_n, t)$ ... some idea]

If $H$ doesn't depend on time, can separate variables:

$$\psi(\vec{r}_1, \vec{r}_2, t) = \psi(\vec{r}_1, \vec{r}_2) e^{-i\frac{E}{\hbar}t}$$
\[
\text{with } -\frac{\hbar^2}{2m} \nabla_i^2 \psi_i + V_1 \psi_i + V_2 \psi_2 = E \psi_i
\]

- If \( V(\vec{r}_1, \vec{r}_2) = V(\vec{r}_1) + V(\vec{r}_2) \)
particles are independent

Can separate further:

\[
\psi(\vec{r}_1, \vec{r}_2) = \psi_1(\vec{r}_1) \psi_2(\vec{r}_2)
\]

\[
-\frac{\hbar^2}{2m} \nabla_1^2 \psi_1 + V_1 \psi_1 = E_1 \psi_1
\]

\[
-\frac{\hbar^2}{2m} \nabla_2^2 \psi_2 + V_2 \psi_2 = E_2 \psi_2
\]

\[
E = E_1 + E_2
\]

Just two copies of single-particle problem

- More interesting when particles interact:
  Typically \( V(\vec{r}_1, \vec{r}_2) = V(\vec{r}_1, -\vec{r}_2) \)
  (no external potential)

Then can't separate \( \psi_1 \) and \( \psi_2 \)

Can separate different variables:
  center-of-mass coordinates

[Problem 5.7]

Define \( \vec{r} = \vec{r}_1 - \vec{r}_2 \) relative coordinate

\[
\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \text{ com coordinate}
\]

Convert Schröd. eqn. to these coordinates
\[ r_2 = r_1 - \mathbf{a} \]
\[ \mathbf{R} = \frac{m_1 r_1 + m_2 (r_1 - \mathbf{a})}{m_1 + m_2} = r_1 - \frac{m_2}{m_1 + m_2} \mathbf{a} \]

So
\[ r_1 = \mathbf{R} - \frac{m_1}{m_1 + m_2} \mathbf{a} \]
\[ r_2 = r_1 - \mathbf{a} = \mathbf{R} + \left(\frac{m_1}{m_1 + m_2} - 1\right) \mathbf{a} \]

\[ r_2 = \mathbf{R} - \frac{m_1}{m_1 + m_2} \mathbf{a} \]

Define \( M = m_1 + m_2 \),

\[ = \text{total mess} \]

Need to convert derivatives \( \frac{d}{dt} \text{ to } \frac{d}{dx} \).

\[ \frac{d}{dx_1} = \frac{d}{dx} \frac{dx_1}{dx} + \frac{d}{dx} \frac{dx_1}{dx} \]
\[ X = \frac{m_1 x_1 + m_2 x_2}{M} \quad x = x_1 - x_2 \]

So
\[ \frac{d}{dx_1} = \frac{m_1}{M} \frac{d}{dx} + \frac{d}{dx} \]

Similarly for other coordinates, set
\[ \frac{d}{dx_1} = \frac{m_1}{M} \frac{d}{dx} + \frac{d}{dx} \]

\[ \frac{d}{dx_2} = \frac{m_2}{M} \frac{d}{dx} - \frac{d}{dx} \]

So
\[ \Delta_1^2 = \frac{m_1^2}{M^2} \Delta_{R_1}^2 + \Delta_{\mathbf{a}}^2 + 2 \frac{m_1}{M} \Delta_{R_1} \cdot \Delta_{\mathbf{a}} \]
\[ \Delta_2^2 = \frac{m_2^2}{M^2} \Delta_{R_2}^2 + \Delta_{\mathbf{a}}^2 - 2 \frac{m_1}{M} \Delta_{R_1} \cdot \Delta_{\mathbf{a}} \]
Put into Schröd. Eqn:

\[
\left[-\frac{1}{2m_1}\nabla_1^2 - \frac{1}{2m_2}\nabla_2^2 + V(r)\right] \phi = E \phi
\]

or

\[
\left[-\frac{1}{2\mu}\nabla_R^2 + \frac{1}{2}(\frac{1}{m_1} + \frac{1}{m_2})\nabla_R^2 + V(r)\right] \phi = E \phi
\]

Define \( \mu = \frac{1}{m_1} + \frac{1}{m_2} \)

\( \mu \) is reduced mass

Then separate:

\[-\frac{1}{2m}\nabla_R^2 \psi = E_R \phi \]

a free particle equation

\[-\frac{1}{2\mu}\nabla_R^2 \psi + V(r) \phi = E \phi \]

a single particle in potential

Total \( E = E_R + E_\phi \)

Usually don't even think about \( \psi \); \( E_\phi \)

Know solutions are plane waves \( \psi \)

You already know how to solve all the single particle problems that have exact solutions

So you can apply that directly to two-body systems

Note: doesn't work if there's an external potential \( V(r) \).
Note, in general, particles will also have spin

\[ 2 \psi (\vec{r}_1, \vec{r}_2, t) \rightarrow 4 \psi (\vec{r}_1, \vec{s}_1, \vec{r}_2, \vec{s}_2, t) \]

Often (not always) the case that spatial and spin parts separate

\[ \rightarrow 2 \psi (\vec{r}_1, \vec{s}_1, t) \chi (\vec{s}_2, \vec{s}_2, t) \]

Then above discussion applies to spatial part

Use techniques from Ch 4 to handle spin part (know how to combine two spins)

**Exchange Symmetry**

Now suppose our particles are the same type — two electrons, for instance.

Then particles are indistinguishable: can't tell one from the other.

Must have \( H(\vec{r}_1, \vec{r}_2) = H(\vec{r}_2, \vec{r}_1) \)

symmetric under particle exchange

Use symmetry to define a new operator that commutes with \( H \):

\[ \mathcal{P} = \text{exchange operator} \]

\[ \mathcal{P} \psi (\vec{r}_1, \vec{r}_2) = \psi (\vec{r}_2, \vec{r}_1) \]
To see that $P$ commutes with $H$:

$$PH^2 = P[H(r_1, r_2) 4(r_1, r_2)] = H(r_1, r_2) 4(r_1, r_2)$$

$$= H(r_1, r_2) 4(r_1, r_2)$$

$$= H 4$$

So $[P, H] = 0$

So we can find eigenstates of $H$ that are also eigenstates of $P$.

What are eigenvalues of $P$?

Have $P^2 4(r_1, r_2) = P 4(r_1, r_2) = 4(r_1, r_2)$

So $P^2 = 1$

Suppose $\phi$ is an eigenstate of $P$.

Then $P\phi = \lambda \phi$

$P^2 \phi = \lambda^2 \phi = \phi$

So $\lambda = \pm 1$

Two eigenstates: $\lambda = 1 \Rightarrow$ even exchange symmetry

$\lambda = -1 \Rightarrow$ odd exchange symmetry

Eigenstates have form

$$4(r_1, r_2) = A\left[2h_a(r_1) 2h_b(r_2) \pm 2h_a(r_2) 2h_b(r_1)\right]$$

+ for even, - for odd