Lecture 15

Last time, calculated relativistic correction to energy levels of hydrogen

$$E_r^{(1)} = -\frac{(E_n)^2}{2mc^2}\left[\frac{4n}{l+\frac{1}{2}} - 3\right]$$

Another effect: spin-orbit coupling

Know that electron has "spin" $\underset{\sim}{S}$

Kind of like to think of electron as "spinning ball of charge"

→ Can't really be right, any mechanically spinning object has integer spin

But we don't really have any better picture.

One consequence:
electrons should have a magnetic moment

Recall from E&M:
Current loop with area A & current I has moment

$$\vec{\mu} = I A \hat{n}$$

$\hat{n}$ = normal to loop

In a spinning ball of charge:

Consider piece dq

Located distance r from axis

Orbits in a circle, encloses area $A = \pi r^2$

If rotational period is T
then current

$$dI = \frac{dq}{T}$$

Gives $d\vec{\mu} = \pi r^2 \frac{dq}{T} \hat{n}$

$\hat{n}$ = axis of rotation

Could calculate total moment as

$$\vec{\mu} = \hat{n} \int \frac{\pi r^2}{T} \rho_Q(\vec{r})\, d^3 r$$

ρ_Q = charge density
$dq = \rho_Q d^3 r$

Of course, we don't know ρ for electron, doubt it even exists.

But can relate μ to something we do know

Look again at dq. Also has mass dm

Since it is spinning, have angular momentum

$$dL = \omega\, dI = \omega r^2 dm \qquad dI = \text{moment of inertia}$$

$$= \frac{2\pi r^2}{T} dm \qquad \omega = \frac{2\pi}{T}$$

$$\vec{L} = \hat{n} \int \frac{2\pi r^2}{T} \rho_m(\vec{r})\, d^3\vec{r} \qquad \rho_m = \text{mass density}$$

If we assume $\rho_m \propto \rho_Q$

then must have $\frac{\rho_Q}{\rho_m} = \frac{q}{m} = \frac{\text{total charge}}{\text{total mass}}$

Then
$$\vec{\mu} = \hat{n}\,\frac{q}{m}\int \frac{\pi r^2}{T}\,\rho_m\,d^3r$$
$$= \hat{n}\,\frac{q}{2m}\int \frac{2\pi r^2}{T}\,\rho_m\,d^3r$$
$$\boxed{\vec{\mu} = \frac{q}{2m}\,\vec{L}}$$

So magnetic moment is related to angular momentum

This holds for any rotating object, including electron with orbital $\vec{L} \neq 0$

Does it work for intrinsic spin?

Sort of. Measurement gives

$$\vec{\mu}_e = -\frac{e}{m}\vec{S} \qquad \text{off by factor of 2.}$$

No way to get with classical physics.
Does come out of relativistic QM,
but that is hard.

Actually, even Dirac theory is slightly wrong,

write $\vec{\mu}_e = -g\,\frac{e}{2m}\,\vec{S}$

$g =$ "g-factor"

QM predicts $g = 2$

Quantum field theory predicts $g = 2.002\ldots$
Can predict out to 12 decimal places

Amazingly, can measure g with same precision
→ Best test of a physical theory in science

So OK. Electrons have a magnetic moment.

Does it matter?

Yes.

In rest frame of electron, proton (charge +e) is orbiting it (if $\vec{L} \neq 0$)

Proton motion forms a current loop: generates a B-field

Magnetic dipole in B-field has energy
$$H' = -\vec{\mu} \cdot \vec{B}$$
→ treat with perturbation theory.

But hold on a second –
Rest frame of electron?
Can we do that?

Two concerns:
First, electron isn't at a well defined position, due to QM.

Actually OK.
Proton isn't at a well defined position either, & it never bothered us.

Really we're working in relative coordinates, $\vec{r}$ = distance from electron to proton.

Even if we don't know absolutely where e^- & proton are, can still think about relative positions.

Second, even classically, if $\vec{L} \neq 0$ then electron is orbiting proton.
= accelerating

Electron rest frame is not inertial.

This does matter. Unfortunately, dealing with it is rather hard.
Read about it in Jackson's Classical Electrodynamics, chapter 11.
(Basically graduate level special relativity)

Net effect called "Thomas precession"

$$H' \to -\frac{g-1}{g}\,\vec{\mu}\cdot\vec{B} \approx -\frac{1}{2}\,\vec{\mu}\cdot\vec{B}$$

Still need to figure out $\vec{B}$:

Get field at center of current loop, radius r from Biot-Savart law

Find $B = \frac{\mu_0 I}{2r}$ $\quad \mu_0$ = permeability of free space, not magnetic moment

Here $I = \frac{+e}{T}$ $\quad T$ = orbital period

Relate to orbital ang. momentum $\vec{L}$:

$$L = 2\pi m r^2 \frac{1}{T}$$

So $B = \frac{\mu_0}{2r}\, e\, \frac{L}{2\pi m r^2}$ $\quad$ use $\mu_0 = \frac{1}{\varepsilon_0 c^2}$

$$\boxed{\vec{B} = \frac{1}{4\pi\varepsilon_0}\,\frac{e}{mc^2}\,\frac{1}{r^3}\,\vec{L}}$$

So we get perturbation

$$H' = -\frac{1}{2}\vec{\mu}\cdot B$$

$$= -\frac{1}{2}\left(-\frac{e}{m}\vec{S}\right)\cdot\left(\frac{1}{4\pi\varepsilon_0}\frac{e}{mc^2}\frac{1}{r^3}\vec{L}\right)$$

$$\boxed{H' = \frac{1}{2}\frac{e^2}{4\pi\varepsilon_0}\frac{1}{m^2c^2r^3}\vec{S}\cdot\vec{L}}$$

Called spin orbit interaction, because it couples $\vec{S}$ and $\vec{L}$

Solve next time

But note:

This is also somehow a relativistic effect.

- Need relativity to get Thomas precession
- Note $\frac{1}{mc^2}$ in H'

  Goes away as $c \to \infty$

Why? Magnetic fields are intrinsically relativistic things, come from moving charges

That's why this and straightforward relativistic correction from last time have similar magnitudes.