Lecture 17

Last time finished fine structure calculation

States labled by (n, l, s, j, m_j)

Energies depend only on n and j. Write nicely

$$\boxed{E_{nj} = -\frac{mc^2\alpha^2}{2n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2} - \frac{3}{4}\right)\right]}$$

$$\alpha = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{\hbar c} \approx \frac{1}{137} \quad \text{(dimensionless)}$$

= Fine structure constant

But that's not all!

Suppose we put atom in an external magnetic field

Energies shift: Zeeman effect

Perturbation is again $-\vec{\mu}\cdot\vec{B}$ here $\vec{B}$ = external field under our control

Now $\vec{\mu}$ has two components

$\vec{\mu}_s$ = intrinsic dipole moment = $-\frac{e}{m}\vec{S}$

$\vec{\mu}_l$ = "classical" moment formed by electron orbiting nucleus = $-\frac{e}{2m}\vec{L}$

These just add

$$\vec{\mu} = \frac{e}{2m}\left(\vec{L} + 2\vec{S}\right)\cdot\vec{B}$$

Define coordinates so z is along $\vec{B}$:

$$H' = \frac{e}{2m}(L_z + 2S_z)B$$

Now in our original solutions, states are eigenstates of L_z and S_z:

$$E^{(1)} = \frac{e\hbar}{2m}B(m_\ell + 2m_s)$$

But fine structure breaks this degeneracy.

Get three cases:

I) $H'_{fs} >> H'_{zeeman}$: Treat H'_{zeeman} as perturbation to fine structure states

II) $H'_{fs} << H'_{zeeman}$: Treat H'_{fs} as perturbation to zeeman states

III) If $H'_{fs} \approx H'_{zeeman}$, treat both as perturbation to plain coulomb potential

Equivalently, compare size of B to size of field produced by "orbiting" proton.

Recall $\vec{B}_{proton} = \frac{1}{4\pi\varepsilon_0}\frac{e}{mc^2}\frac{1}{r^3}\vec{L}$

estimate $r \to a$, $L \to \hbar$

$$B_{proton} = \frac{1}{4\pi\varepsilon_0}\frac{e}{mc^2}\frac{\hbar}{a^3}$$

Plug in numbers, ≈ 12 Tesla

That is pretty big. About as big as we can produce

Treat $B \ll B_{proton}$ case today

Want $E^{(1)} = \frac{eB}{2m} \langle L_z + 2S_z \rangle$

Note $L_z + S_z = J_z$, and $\langle J_z \rangle = m_j$

Still leaves $\langle S_z \rangle$

Working out for real is a pain ... see Supp'l. 3 online.

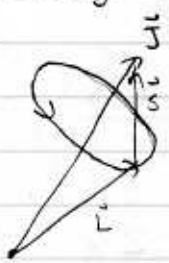
But a nice "geometrical" method also works:

Imagine that in $|j\, m_j\rangle$ state, $\vec{J}$ points in some direction, we just can't say what it is.

Because $\vec{J}$ is conserved

But $\vec{S}$ (and $\vec{L}$) are not conserved.

Imagine $\vec{S}$ precessing rapidly around $\vec{J}$



So part of $\vec{S}$ that is $\perp$ to $\vec{J}$ is rapidly oscillating $\to 0$ on average.

Projection of $\vec{S}$ that is $\parallel \vec{J}$ is

$$\vec{S}_{\parallel} = (\vec{S} \cdot \hat{J})\, \hat{J} \qquad \hat{J} = \frac{\vec{J}}{|\vec{J}|}$$

$$\to \frac{\vec{S} \cdot \vec{J}}{J^2}\, \vec{J}$$

So use $\langle S_z \rangle = \left\langle \frac{\vec{S}\cdot\vec{J}}{J^2} J_z \right\rangle = \frac{1}{\hbar} \frac{m_j}{j(j+1)} \langle \vec{S}\cdot\vec{J} \rangle$

To get $\vec{S}\cdot\vec{J}$, use $\vec{L} = \vec{J} - \vec{S}$

$L^2 = J^2 + S^2 - 2\vec{J}\cdot\vec{S}$

$$\text{So } \langle \vec{J}\cdot\vec{S} \rangle = \frac{1}{2}\langle J^2 + S^2 - L^2 \rangle$$

$$= \frac{\hbar^2}{2}\left[j(j+1) + s(s+1) - \ell(\ell+1) \right]$$

$$\text{So } \langle S_z \rangle = \pm m_j \frac{j(j+1) - \ell(\ell+1) + \frac{3}{4}}{2j(j+1)}$$

$$\text{and } E^{(1)} = \frac{eB}{2m}\langle J_z + S_z \rangle = \frac{eB\hbar}{2m} m_j \left(1 + \frac{j(j+1) - \ell(\ell+1) + \frac{3}{4}}{2j(j+1)} \right)$$

Typically write $E^{(1)} = \mu_B g_J m_j B$

μ_B = Bohr magneton $= \frac{e\hbar}{2m} = 9.27 \times 10^{-24} \frac{J}{T}$

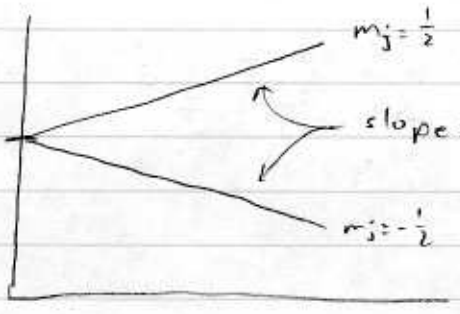
g_J = Landé g-factor

$$= 1 + \frac{j(j+1) - \ell(\ell+1) + \frac{3}{4}}{2j(j+1)}$$

If $\ell = 0$, $j = 1/2$, $g_J = 1 + \frac{3/2}{3/2} = 2$

= g-factor of electron

So ground state of hydrogen acts like:



Separation = Zeeman splitting