Lecture 18

Last time, talked about Zeeman effect:
Energy shift of atom in magnetic field B

If $B \ll B_{internal}$, found $E_Z^{(1)} = \mu_B g_J B m_j$

$$\mu_B = \frac{e\hbar}{2m} \quad : \text{Bohr magneton}$$

$$g_J = 1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)} \quad : \text{Landé g-factor}$$

Note, this is why m is called "magnetic quantum #"
→ magnetic field lifts degeneracy in m

Compare classical: $E = -\mu B \cos\theta$

quantum $E = -\mu B m_j$ $\quad \mu \equiv -\mu_B g_J$ effective moment

$m_j \approx$ quantized $\cos\theta$

Today, consider stronger fields:

If $B \gg B_{internal}$, work in uncoupled (m_l, m_s) basis

Note: Need μB small compared to $|E_n - E_{n+1}|$, still a perturbation w/respect to Bohr states.

Recall $H'_Z = \frac{e}{2m} B (L_z + 2S_z)$

$$\boxed{E_Z^{(1)} = \mu B (m_l + 2m_s)}$$

Now apply fine structure as a perturbation

But Zeeman effect has broken degeneracy.
Just use nondegenerate perturbation theory

Relativistic term $H'_r = -\frac{p^4}{8m^3c^2}$

We calculated this in (m_ℓ, m_s) basis before...
nothing has changed here, so

$$E_r^{(1)} = -\frac{(E_n)^2}{2mc^2}\left[\frac{4n}{\ell+1/2} - 3\right]$$

Spin-orbit effect was harder

$$H' = \frac{1}{2}\frac{e^2}{4\pi\varepsilon_0}\frac{1}{m^2c^2r^3}\vec{S}\cdot\vec{L}$$

Before, had to use J, m_j to deal with $\vec{S}\cdot\vec{L}$
because of degeneracy.

Now we can just evaluate

$$\langle m_\ell\, m_s | \vec{S}\cdot\vec{L} | m_\ell\, m_s\rangle$$

$$= \underbrace{\langle S_x\rangle\langle L_x\rangle + \langle S_y\rangle\langle L_y\rangle}_{=0 \text{ for eigenstates of } S_z,\, L_z} + \underbrace{\langle S_z\rangle\langle L_z\rangle}_{\hbar^2 m_s m_\ell}$$

Spin-orbit effect becomes

$$E_{so}^{(1)} = \frac{1}{2}\frac{e^2}{4\pi\varepsilon_0}\frac{\hbar^2}{m^2c^2r^3}\frac{m_s m_\ell}{\ell(\ell+\frac{1}{2})(\ell+1)\,n^3a^3}$$

$$= \frac{(E_n)^2}{mc^2}\frac{2n\, m_s m_\ell}{\ell(\ell+\frac{1}{2})(\ell+1)}$$

Combine, get

$$E_{fs}^{(1)} = \frac{(E_n)^2}{2mc^2}\left[\frac{4n\, m_s m_\ell}{\ell(\ell+\frac{1}{2})(\ell+1)} - \frac{4n}{\ell+\frac{1}{2}} + 3\right]$$

$$\boxed{E_{fs}^{(1)} = -\frac{E_1}{n^3}\alpha^2\left\{\frac{3}{4n} - \left[\frac{l(l+1) - m_l m_s}{l(l+\frac{1}{2})(l+1)}\right]\right\}}$$

Total energy is $E_n + E_z^{(1)} + E_{fs}^{(1)}$

So what if $B \approx B_{int}$?
That's the hardest case.

Need to treat Zeeman & fine structure on equal footing.
Can't avoid doing real degenerate PT

Can't solve general problem. Just do $n=2$ levels as example.

Work in $|j\, m\rangle$ basis. $|m_l\, m_s\rangle$ basis also works, but its a bit harder

Need to relate $|j\, m\rangle$ and $|l\, m_l\rangle|s\, m_s\rangle$ states

Use Clebsh-Gordon coefficients. Hoorey!

Have 8 states:

$$\qquad\qquad\qquad\qquad\qquad\qquad j\ m_j \qquad\quad l\ m_l \quad s\ m_s$$

ψ_1: $l=0,\ j=\frac{1}{2},\ m_j=\frac{1}{2}$: $\quad |\tfrac{1}{2}\ \tfrac{1}{2}\rangle = |0\,0\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$

ψ_2: $l=0,\ j=\frac{1}{2},\ m_j=-\frac{1}{2}$ $\quad |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = |0\,0\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

ψ_3: $l=1,\ j=\frac{3}{2},\ m_j=\frac{3}{2}$ $\quad |\tfrac{3}{2}\ \tfrac{3}{2}\rangle = |1\,1\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$

ψ_4: $l=1,\ j=\frac{3}{2},\ m_j=-\frac{3}{2}$ $\quad |\tfrac{3}{2}\ -\tfrac{3}{2}\rangle = |1\,{-1}\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

ψ_5: $l=1,\ j=\frac{3}{2},\ m_j=\frac{1}{2}$ $\quad |\tfrac{3}{2}\ \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|1\,0\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,|1\,1\rangle|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

coefficients from table 4.8

ψ_6 : $l=1$, $j=\frac{1}{2}$, $m_j=\frac{1}{2}$: $|\frac{1}{2}\ \frac{1}{2}\rangle = -\sqrt{\frac{1}{3}}\,|1\,0\rangle|\frac{1}{2}\ \frac{1}{2}\rangle + \sqrt{\frac{2}{3}}\,|1\,1\rangle|\frac{1}{2}\ -\frac{1}{2}\rangle$

ψ_7 : $l=1$, $j=\frac{3}{2}$, $m_j=-\frac{1}{2}$: $|\frac{3}{2}\ -\frac{1}{2}\rangle = \sqrt{\frac{1}{3}}\,|1\,-1\rangle|\frac{1}{2}\ \frac{1}{2}\rangle + \sqrt{\frac{2}{3}}\,|1\,0\rangle|\frac{1}{2}\ -\frac{1}{2}\rangle$

ψ_8 : $l=1$, $j=\frac{1}{2}$, $m_j=-\frac{1}{2}$: $|\frac{1}{2}\ -\frac{1}{2}\rangle = -\sqrt{\frac{2}{3}}\,|1\,-1\rangle|\frac{1}{2}\ \frac{1}{2}\rangle + \sqrt{\frac{1}{3}}\,|1\,0\rangle|\frac{1}{2}\ -\frac{1}{2}\rangle$

Work out W matrix in this basis

Fine structure part is diagonal, since we used $(j\,m)$ basis

$$\langle j\,m|H'_{fs}|j\,m\rangle = -|E_1|\frac{\alpha^2}{2^4}\left(\frac{2}{j+\frac{1}{2}} - \frac{3}{4}\right)$$

$$\text{if } j=\tfrac{1}{2}: \ -|E_1|\frac{\alpha^2}{2^4}\left(2-\frac{3}{4}\right) = -|E_1|\frac{\alpha^2}{2^4}\frac{5}{4} = -\frac{5}{64}\alpha^2|E_1|$$

$$\text{if } j=\tfrac{3}{2}: \ -|E_1|\frac{\alpha^2}{2^4}\left(1-\frac{3}{4}\right) = -\frac{1}{64}\alpha^2|E_1|$$

Define $\gamma = \frac{\alpha^2}{64}|E_1|$

Then $\langle j\,m|H'_{fs}|j\,m\rangle = -\gamma$ for $j=\frac{3}{2}$
-5γ for $j=\frac{1}{2}$