Lecture 18

Last time, talked about Zeeman effect.

Energy shift of atom in magnetic field $B$

If $B \ll$ Binternal, found $E_2 = \mu_B g_J B m_J$

$$\mu_B = \frac{e}{2m}$$ Bohr magneton.

$$g_J = 1 + \frac{j(j+1)-8(j+1)+4}{2j(2j+1)}$$ Landé g-factor.

Note, this is why $m$ is called "magnetic quantum number" - magnetic field lifts degeneracy in $m$.

Compare classical: $E = -\mu_B B \cos \theta$

Quantum $E = -\mu_B B m_J$ $m_J = \mu_B g_J$ effective moment.

Today, consider stronger fields:

If $B \gg$ Binternal, work in uncoupled $(m_2, m_3)$ basis.

Recall $H' = \frac{e}{2m} B \left( L^2 + 2S_z \right)$

$$E_2^{(n)} = \mu_B (m_2 + 2m_3)$$

Note: The $\mu_B$ small compared to $|eB|$, still a perturbation w/respect to Bohr states.

Now apply fine structure as a perturbation.

But Zeeman effect has broken degeneracy, just use non-degenerate perturbation theory.
Relativistic term \( H'_r = -\frac{p^2}{8m^2c^2} \)

We calculated this in \((m, n, l)\) basis before, nothing has changed here, so

\[ E^{(l)} = \frac{|E_n|^2}{2mc^2} \left[ l(l+1) - 3 \right] \]

Spin-orbit effect was harder

\[ H'_s = \frac{\hbar^2}{8\pi^2 \hbar^2} \frac{e^2}{m^2c^2 \hbar^2} S \cdot L \]

Before, had to use \( J, m_J\) to deal with \( S \cdot L\) because of degeneracy.

Now we can just evaluate

\[ \langle s_m, m_s | S \cdot L | m_s, m_s \rangle \]

\[ = \langle S_x \rangle \langle L_x \rangle + \langle S_y \rangle \langle L_y \rangle + \langle S_z \rangle \langle L_z \rangle \]

\[ = 0 \text{ for eigenstates } S_x, L_z \]

Spin-orbit effect becomes

\[ E^{(l)}_{s0} = \frac{1}{2} \frac{e^2}{4\pi^2 \hbar^2 m^2c^2} \frac{\hbar^2}{m_0^2} \frac{m_n}{\hbar \sqrt{m_0^2}} \left( l(l+1) \right)^{1/2} \frac{a^3}{n^3} \]

\[ = \frac{|E_n|^2}{mc^2} \left( \frac{2n(m_0m_n)}{l(l+1)(l+1)} \right) \]

Combine, get

\[ E^{(l)}_{s0} = \frac{1}{2mc^2} \left[ \frac{4n(m_0m_n)}{l(l+1)(l+1)} - \frac{4n}{l+1} + 3 \right] \]
\[ E^{(x)} = -\frac{E_1}{n^3} \alpha^2 \left[ \frac{3}{4n} \right] - \left( \frac{\ell (\ell + 1) - m - 1}{\ell (\ell + 1) (\ell + 2)} \right) \frac{\ell}{3} \]

Total energy is \( E_n + E_2 + E_{\psi} \)

So what if \( B \neq 0 \)?
That's the hardest case.

Need to treat Zeeman + fine structure on equal footing.
Can't avoid doing real degenerate PT

Can't solve general problem. Just do \( n = 2 \) levels as example.

Work in \(|j,m\rangle\) basis. \( |m_\alpha m_\beta \rangle\) basis also works, but it's a bit harder.

Need to relate \(|j,m\rangle\) and \(|lm_\alpha \rangle|s \rangle|m_\beta \rangle\) states.

Use Clebsch-Gordan coefficients. Hooray!

Have 8 states:

\[ |j = 0, m = \pm 1, m_s = \pm \frac{1}{2} \rangle = \frac{1}{\sqrt{2}} |00\rangle \frac{1}{2} \pm \frac{1}{2} \]

\[ |j = 0, m = \pm 1, m_s = \mp \frac{1}{2} \rangle = \frac{1}{\sqrt{2}} |00\rangle \pm \frac{1}{2} \mp \frac{1}{2} \]

\[ |j = 1, m = \mp \frac{1}{2}, m_s = \pm \frac{3}{2} \rangle = |11\rangle \frac{1}{2} \pm \frac{1}{2} \]

\[ |j = 1, m = \pm \frac{3}{2}, m_s = \mp \frac{3}{2} \rangle = |11\rangle \frac{1}{2} \mp \frac{1}{2} \]

\[ |j = 1, m = \pm \frac{1}{2}, m_s = \pm \frac{5}{2} \rangle = \frac{1}{\sqrt{2}} \left( |10\rangle \frac{1}{2} \pm \frac{1}{2} \right) \pm \frac{1}{2} \]

Coefficients from Table 4.8
\[ l = 1, j = \frac{1}{2}, m_j = \frac{1}{2} \]
\[ \frac{1}{2} > = - \frac{\sqrt{5}}{15} |10> |\frac{1}{2} > + \frac{\sqrt{5}}{15} |11> |\frac{1}{2} < \]

\[ l = 1, j = \frac{7}{2}, m_j = -\frac{3}{2} \]
\[ \frac{7}{2} > = - \frac{\sqrt{5}}{15} |11> |\frac{1}{2} > + \frac{\sqrt{5}}{15} |10> |\frac{1}{2} < \]

\[ l = 1, j = \frac{5}{2}, m_j = -\frac{5}{2} \]
\[ \frac{5}{2} > = - \frac{\sqrt{5}}{15} |11> |\frac{1}{2} > + \frac{\sqrt{5}}{15} |10> |\frac{1}{2} < \]

Work out W matrix in this basis.

Fine structure part is diagonal, since we used (j, m) basis.

\[ \langle j_m - 1 \mid H_{\text{fs}} \mid j_m \rangle = - |E| \frac{\alpha^2}{2^4} \left( \frac{1}{2} - \frac{3}{4} \right) \]

If \( j = \frac{1}{2} \):  \(- |E| \frac{\alpha^2}{2^4} \left( 2 - \frac{3}{4} \right) = - |E| \frac{\alpha^2}{2^4} \frac{5}{4} = \frac{5}{64} \frac{\alpha^2}{|E|} \)

If \( j = \frac{3}{2} \):  \(- |E| \frac{\alpha^2}{2^4} \left( 1 - \frac{3}{4} \right) = - \frac{1}{64} \frac{\alpha^2}{|E|} \)

Define \( \gamma = \frac{\alpha^2}{64} |E| \)

Then \[ \langle j_m - 1 \mid H_{\text{fs}} \mid j_m \rangle = - \gamma \quad \text{for } j = \frac{3}{2} \]
\[ -5 \gamma \quad \text{for } j = \frac{1}{2} \]