Lecture 19

Working on intermediate field Zeeman effect

$$H' = H'_{fs} + H'_z$$

Need to use real degenerate PT

Choose basis: $|n l s j m_j\rangle$ states, with $n=2$

Generally, can write

$$|l s j m_j\rangle = \sum_{m_l, m_s} C^{l s j}_{m_l m_s m_j} |l m_l\rangle |s m_s\rangle$$

coefficients C = Clebsh - Gordon coefficients

See Griffiths § 4.4.3

Note, since $J_z = L_z + S_z$, need $m_j = m_s + m_l$
(otherwise $C = 0$)

Also, l is unchanged, so no coupling between $l=0$ and $l=1$ states.

Listed cases last time. Only nontrivial ones are

$$\psi_5 = |1\ \tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\, |1\,0\rangle |\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\, |1\,1\rangle |\tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle$$

$$\psi_6 = |1\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle = -\sqrt{\tfrac{1}{3}}\, |1\,0\rangle |\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\, |1\,1\rangle |\tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle$$

$$\psi_7 = |1\ \tfrac{1}{2}\ \tfrac{3}{2}\ {-\tfrac{1}{2}}\rangle = \sqrt{\tfrac{1}{3}}\, |1\,{-1}\rangle |\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\, |1\,0\rangle |\tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle$$

$$\psi_8 = |1\ \tfrac{1}{2}\ \tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle = -\sqrt{\tfrac{2}{3}}\, |1\,{-1}\rangle |\tfrac{1}{2}\ \tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\, |1\,0\rangle |\tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle$$

Need to get W matrix

Fine structure part is easy, since it is diagonal

$$\langle j\, m | H'_{fs} | j\, m \rangle = E_1 \frac{\alpha^2}{2^4}\left(\frac{2}{j+\frac{1}{2}} - \frac{3}{4}\right)$$

$$\text{if } j = \tfrac{1}{2}, \quad = E_1 \frac{\alpha^2}{2^4}\left(2 - \frac{3}{4}\right) = E_1 \frac{\alpha^2}{2^4} \frac{5}{4}$$

$$= \frac{5}{64} \alpha^2 E_1$$

$$\text{if } j = \tfrac{3}{2}, \quad = E_1 \frac{\alpha^2}{2^4}\left(1 - \frac{3}{4}\right) = \frac{1}{64} \alpha^2 E_1$$

Define $\gamma = \frac{\alpha^2}{64} |E_1|$

$$\langle j m | H'_{fs} | j m \rangle = \begin{cases} -\gamma & \text{for } j = \frac{3}{2} \\ -5\gamma & \text{for } j = \frac{1}{2} \end{cases}$$

But H_z' is not diagonal $\quad H_z' = \frac{e}{2m} B (J_z + S_z)$

$$\langle j' m' | H_z' | j m \rangle = \mu_B B\, m\, \delta_{mm'} + \frac{e}{2m} B \langle j' m' | S_z | j m \rangle$$

Diagonal terms:

$$\langle \psi_1 | S_z | \psi_1 \rangle = \langle 0 0 | \langle \tfrac{1}{2} \tfrac{1}{2} | \; S_z \; | 0 0 \rangle | \tfrac{1}{2} \tfrac{1}{2} \rangle$$

$$= \langle \tfrac{1}{2} \tfrac{1}{2} | S_z | \tfrac{1}{2} \tfrac{1}{2} \rangle = \frac{\hbar}{2}$$

Similar for ψ_2, ψ_3, ψ_4

$$\langle \psi_5 | S_z | \psi_5 \rangle = \left(\sqrt{\tfrac{2}{3}} \langle 1\,0| \langle \tfrac{1}{2}\,\tfrac{1}{2}| + \sqrt{\tfrac{1}{3}} \langle 1\,1| \langle \tfrac{1}{2}\,-\tfrac{1}{2}| \right) S_z \left(\sqrt{\tfrac{2}{3}}\, |1\,0\rangle |\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\, |1\,1\rangle |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle \right)$$

$$= \tfrac{2}{3} \langle \tfrac{1}{2}\,\tfrac{1}{2} | S_z | \tfrac{1}{2}\,\tfrac{1}{2} \rangle + \tfrac{1}{3} \langle \tfrac{1}{2}\,-\tfrac{1}{2} | S_z | \tfrac{1}{2}\,-\tfrac{1}{2} \rangle$$

(since S_z doesn't change L state)

$$= \tfrac{2}{3} \cdot \tfrac{\hbar}{2} + \tfrac{1}{3}\left(-\tfrac{\hbar}{2}\right) = \tfrac{1}{6}\hbar$$

Similarly for ψ_6, ψ_7, ψ_8

$$\langle \psi_6 | S_z | \psi_6 \rangle = \tfrac{1}{3} \cdot \tfrac{1}{2} + \tfrac{2}{3}\left(-\tfrac{1}{2}\right) = -\tfrac{\hbar}{6}$$

$$\langle \psi_7 | S_z | \psi_7 \rangle = -\tfrac{\hbar}{6}$$

$$\langle \psi_8 | S_z | \psi_8 \rangle = \tfrac{\hbar}{6}$$

For off diagonals, only get coupling between states 5, 6, 7, 8

$$\langle \psi_6 | S_z | \psi_5 \rangle = \left(-\sqrt{\tfrac{1}{3}} \langle 1\,0| \langle \tfrac{1}{2}\,\tfrac{1}{2}| + \sqrt{\tfrac{2}{3}} \langle 1\,1| \langle \tfrac{1}{2}\,-\tfrac{1}{2}| \right) S_z \left(\sqrt{\tfrac{2}{3}}\, |1\,0\rangle |\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\, |1\,1\rangle |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle \right)$$

$$= -\tfrac{\sqrt{2}}{3} \langle \tfrac{1}{2}\,\tfrac{1}{2} | S_z | \tfrac{1}{2}\,\tfrac{1}{2} \rangle + \tfrac{\sqrt{2}}{3} \langle \tfrac{1}{2}\,-\tfrac{1}{2} | S_z | \tfrac{1}{2}\,-\tfrac{1}{2} \rangle$$

$$= -\tfrac{\sqrt{2}}{3}\,\tfrac{\hbar}{2} - \tfrac{\sqrt{2}}{3}\,\tfrac{\hbar}{2} = \tfrac{\sqrt{2}}{3}\hbar$$

Get rest with similar procedure

Get W matrix:

$$(-1)\times\begin{bmatrix} 5\gamma-\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5\gamma+\beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma-2\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma+2\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma-\frac{2}{3}\beta & \frac{\sqrt{2}}{3}\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{3}\beta & 5\gamma-\frac{1}{3}\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma+\frac{2}{3}\beta & \frac{\sqrt{2}}{3}\beta \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{3}\beta & 5\gamma+\frac{1}{3}\beta \end{bmatrix}$$

$$\beta = \mu_B B$$

See that it is mostly diagonal
Just two 2×2 blocks

Diagonalize each seperately
for instance:

$$\begin{vmatrix} \gamma-\frac{2}{3}\beta-\lambda & \frac{\sqrt{2}}{3}\beta \\ \frac{\sqrt{2}}{3}\beta & 5\gamma-\frac{1}{3}\beta-\lambda \end{vmatrix} = 0$$

Solve, get $\lambda_\pm = -3\gamma + \frac{\beta}{2} \pm \sqrt{4\gamma^2 + \frac{2}{3}\gamma\beta + \frac{\beta^2}{4}}$

gives energy of states for all values of B

Other block gives $\lambda = -3\gamma - \frac{\beta}{2} \pm \sqrt{4\gamma^2 - \frac{2}{3}\gamma\beta + \frac{\beta^2}{4}}$

Actual formulas aren't that important, but its good to understand the procedure.

Very briefly, cover last topic: hyperfine effect

Recall proton is also spin $\frac{1}{2}$

It has a magnetic moment, just like electron

Griffiths uses $\vec{S}_p = \vec{I}$

Write $\mu_p = \frac{e}{2m_p} g_p \vec{I}$ $\qquad \vec{I}$ = proton spin

$g_p = 5.59$: proton has complicated internal structure

Moments of electron and proton interact

Get energy from E&M:

$$H'_{hf} = \frac{\mu_0 g_p e^2}{8\pi m_p m_e} \frac{[3(\vec{I}\cdot\hat{r})(S\cdot\hat{r}) - \vec{I}\cdot\vec{S}]}{r^3} + \frac{\mu_0 g_p e^2}{3m_p m_e} \vec{I}\cdot\vec{S}\,\delta^3(\vec{r})$$

Pretty messy. Turns out that for $l=0$ states, first term → 0

Consider $n=1$ state

$$\langle \psi_{100} | H'_{hf} | \psi_{100} \rangle = \frac{\mu_0 g_p e^2}{3m_p m_e} \underbrace{|\psi_{100}(0)|^2}_{\frac{1}{\pi a^3}} \langle \vec{I}\cdot\vec{S} \rangle$$

Need to handle dot product.

Griffiths uses $\vec{S} = \vec{F}$

Standard trick: define total spin $\vec{F} = \vec{I} + \vec{S}$

$$\vec{I}\cdot\vec{S} = \frac{1}{2}(F^2 - S^2 - I^2)$$

Have $F=0$ or $F=1$

if $F=0$, $\vec{I}\cdot\vec{S} = \frac{1}{2}\left(0 - \frac{3}{4} - \frac{3}{4}\right) = -\frac{3}{4}$

if $F=1$, $\vec{I}\cdot\vec{S} = \frac{1}{2}\left(2 - \frac{3}{4} - \frac{3}{4}\right) = \frac{1}{4}$

So energies are

$$E_{hf}^{(1)} = \frac{4 g_p \hbar^4}{3 m_p m_e^2 c^2 a^4} \times \begin{cases} 1/4 & F=1 \\ -3/4 & F=0 \end{cases}$$

Plug in numbers, $E_{hf}^{(1)} = 5.88 \times 10^{-6}\ eV = h \cdot 1.42\ GHz$