Lecture 2 - Exchange Symmetry

Last time, introduced idea of many-body wavefunctions

$\psi(\vec{r}_1, \vec{r}_2, \dots \vec{r}_N, t)$ = wave function for N particles

$N=2, \quad \psi(\vec{r}_1, \vec{r}_2, t)$

Satisfy Schr. eqn $\quad H\psi = E\psi$

$$H(\vec{r}_1, \vec{r}_2, t) = -\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 + U(\vec{r}_1, \vec{r}_2, t)$$

Independent particles: separable solutions $\psi_1(\vec{r}_1, t)\psi_2(\vec{r}_2, t)$

Interacting particles: separate in center-of-mass coords
$\psi_R(\vec{R}, t)\,\psi_r(\vec{r}, t)$

If particles are identicle, define exchange operator

$$P\,\psi(\vec{r}_1, \vec{r}_2) = \psi(\vec{r}_2, \vec{r}_1)$$

Then $[P, H] = 0$

$\Rightarrow$ H & P have common eigenstates

Eigenvalues of P are $\lambda = \pm 1$

Suppose two single-particle states $\psi_a(\vec{r}),\ \psi_b(\vec{r})$

Parity eigenstates are

$$\psi_\pm(\vec{r}_1, \vec{r}_2) = A\left[\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) \pm \psi_a(\vec{r}_2)\psi_b(\vec{r}_1)\right]$$

For either case, can't say which particle is in which state ... makes sense.

More surprising:

Nature requires that <u>all</u> particles of the same type must be in an exchange eigenstate

→ New thing! No similar requirement for a single particle to be in an energy eigenstate

Furthermore, state is determined by spin of particle:

Integer spin ≡ bosons ≡ positive exchange
$\frac{1}{2}$-integer spin ≡ fermions ≡ negative exchange

Bosons: photons, mesons, + composite particles
Fermions: electrons, protons, neutrons, + composite particles

Most important consequence:
no two fermions can be in the same state

If $\psi_a = \psi_b$, then $\psi_-(\vec{r}_1, \vec{r}_2) = A[\psi_a(\vec{r}_1)\psi_a(\vec{r}_2) - \psi_a(\vec{r}_1)\psi_b(\vec{r}_2)]$
= non-normalizable

Gives Pauli exclusion principle for electrons
→ chemistry → biology → us

<u>Why is this true?</u>

Not just a new rule...
Can be derived from quantum field theory
Can get relation between rotation & exchange properties of quantum fields

But not very satisfying

What does it mean?

→ Identical quantum particles are fundamentally <u>indistinguishable</u>

No way to tell one electron from another

Do a problem: 5.4

a) What is normalization constant A in $\psi_\pm$ if ψ_a & ψ_b are normalized + orthogonal?

Want $\int |\psi_\pm(r_1, r_2)|^2 \, d^3r_1 \, d^3r_2 = 1$

$$\int |A|^2 \left[\psi_a(r_1)\psi_b(r_2) \pm \psi_a(r_2)\psi_b(r_1)\right]\left[\psi_a^*(r_1)\psi_b^*(r_2) \pm \psi_a^*(r_2)\psi_b^*(r_1)\right] d^3r_1 \, d^3r_2$$

$$= |A|^2 \int |\psi_a(r_1)|^2 |\psi_b(r_2)|^2 \pm \psi_a(r_2)\psi_b(r_1)\psi_a^*(r_1)\psi_b^*(r_2) \pm \psi_a(r_1)\psi_b(r_2)\psi_a^*(r_2)\psi_b^*(r_1) + |\psi_a(r_2)|^2 |\psi_b(r_1)|^2 \quad d^3r_1 \, d^3r_2$$

$$= |A|^2 \Big[\int |\psi_a(r_1)|^2 d^3r_1 \int |\psi_b(r_2)|^2 d^3r_2$$
$$+ \int |\psi_b(r_1)|^2 d^3r_1 \int |\psi_a(r_2)|^2 d^3r_2$$
$$\pm \int \psi_b(r_1)\psi_a^*(r_1) \, d^3r_1 \int \psi_a(r_2)\psi_b^*(r_2) \, d^3r_2 \nearrow 0$$
$$\pm \int \psi_a(r_1)\psi_b^*(r_1) \, d^3r_1 \int \psi_b(r_2)\psi_a^*(r_2) \, d^3r_2 \Big] \nearrow 0$$

$= 2|A|^2$ by normalization + orthogonality.

So up to overall phase, $A = \frac{1}{\sqrt{2}}$

For fun, do again with Dirac notation

$$|\psi_+\rangle = A\left(|\psi_a\rangle_1 |\psi_b\rangle_2 \pm |\psi_b\rangle_1 |\psi_a\rangle_2 \right)$$

Want $\langle \psi_+ | \psi_+ \rangle = 1$

$$|A|^2 \left[\left({}_1\langle \psi_a | \, {}_2\langle \psi_b | \pm {}_1\langle \psi_b | \, {}_2\langle \psi_a | \right) \left(|\psi_a\rangle_1 |\psi_b\rangle_2 \pm |\psi_b\rangle_1 |\psi_a\rangle_2 \right) \right]$$

$$|A|^2 \left[{}_1\langle \psi_a | \psi_a \rangle_1 \, {}_2\langle \psi_b | \psi_b \rangle_2 \pm {}_1\langle \psi_a | \psi_b \rangle_1 \, {}_2\langle \psi_b | \psi_a \rangle_2 \right.$$

$$\left. \pm {}_1\langle \psi_b | \psi_a \rangle_1 \, {}_2\langle \psi_a | \psi_b \rangle_2 + {}_1\langle \psi_b | \psi_b \rangle_1 \, {}_2\langle \psi_a | \psi_a \rangle_2 \right]$$

$$= 2|A|^2$$ a bit easier to write

Part (b): What if $\psi_a = \psi_b$?

Only works for bosons

$$|\psi_+\rangle = 2A \, |\psi_a\rangle_1 |\psi_a\rangle_2$$

$$\langle \psi_+ | \psi_+ \rangle = 1 = 4|A|^2 \, {}_1\langle \psi_a | \psi_a \rangle_1 \, {}_2\langle \psi_a | \psi_a \rangle_2$$

So $A = \frac{1}{2}$

$$|\psi_+\rangle = |\psi_a\rangle_1 |\psi_a\rangle_2$$

or $\psi_+(r_1, r_2) = \psi_a(r_1) \psi_a(r_2)$

Exchange Forces

Another way to see effect of symmetry

Suppose particles in states $\psi_a(x)$, $\psi_b(x)$ $\left(\begin{array}{l}\text{orthog.}\\ \text{\& normalized}\end{array}\right)$

Work out $\quad \Delta^2 \equiv \langle (x_1 - x_2)^2 \rangle$ ~ mean square distance between particles

$$= \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2\langle x_1 x_2 \rangle$$

Dirac Notation?

Case I: Distinguishable particles

$$\psi(x_1, x_2) = \psi_a(x_1)\,\psi_b(x_2)$$

$$\langle x_1^2 \rangle = \int x_1^2\, \psi_a(x_1)\psi_b(x_2)\psi_a^*(x_1)\psi_b^*(x_2)\, dx_1$$
$$= \int x_1^2 |\psi_a(x_1)|^2 dx_1 \int |\psi_b(x_2)|^2 dx_2 = \langle x^2 \rangle_a$$

$$\langle x_2^2 \rangle = \langle x^2 \rangle_b$$

$$\langle x_1 x_2 \rangle = \int x_1 |\psi_a(x_1)|^2 dx_1 \int x_2 |\psi_b(x_2)|^2 dx_2 = \langle x \rangle_a \langle x \rangle_b$$

$$\Delta^2 = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b$$

Case II: Identical particles

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_a(x_1)\psi_b(x_2) \pm \psi_b(x_1)\psi_a(x_2)\right]$$

Now

$$\langle x_1^2 \rangle = \frac{1}{2}\Big[\int x_1^2 |\psi_a(x_1)|^2 |\psi_b(x_2)|^2 dx_1 dx_2$$
$$+ \int x_1^2 |\psi_a(x_2)|^2 |\psi_b(x_1)|^2 dx_1 dx_2$$
$$\pm \int x_1^2 \psi_a(x_1)\psi_b^*(x_1)\, dx_1 \int \psi_a^*(x_2)\psi_b(x_2)\, dx_2$$
$$\pm \int x_1^2 \psi_a^*(x_1)\psi_b(x_1)\, dx_1 \int \psi_a(x_2)\psi_b^*(x_2)\, dx_2 \Big]$$

$$= \frac{1}{2}\left(\langle x^2 \rangle_a + \langle x^2 \rangle_b\right)$$

Of course $\langle x_2^2 \rangle = \langle x_1^2 \rangle$ by symmetry

Finally

$$\langle x_1 x_2 \rangle = \frac{1}{2}\Big[\int x_1 x_2 \, |\psi_a(x_1)|^2 |\psi_b(x_2)|^2 \, dx_1 dx_2$$

$$+ \int x_1 x_2 \, |\psi_b(x_1)|^2 |\psi_a(x_2)|^2 \, dx_1 dx_2$$

$$\pm \int x_1 x_2 \, \psi_a(x_1) \psi_b(x_2) \, \psi_b^*(x_1) \psi_a^*(x_2) \, dx_1 dx_2$$

$$\pm \int x_1 x_2 \, \psi_a^*(x_1) \psi_b^*(x_2) \, \psi_b(x_1) \psi_a(x_2) \, dx_1 dx_2 \Big]$$

$$= \frac{1}{2}\Big[\int x_1 |\psi_a|^2 dx_1 \int x_2 |\psi_b|^2 dx_2$$

$$+ \int x_1 |\psi_b|^2 dx_1 \int x_2 |\psi_a|^2 dx_2$$

$$\pm \int x_1 \psi_a \psi_b^* dx_1 \int x_2 \psi_a^* \psi_b \, dx_2$$

$$\pm \int x_1 \psi_a^* \psi_b \, dx_1 \int x_2 \psi_a \psi_b^* dx_2 \Big]$$

$$= \frac{1}{2}\Big[\langle x \rangle_a \langle x_b \rangle + \langle x \rangle_a \langle x_b \rangle$$

$$\pm \langle x \rangle_{ab} \langle x \rangle_{ab}^* \pm \langle x \rangle_{ab}^* \langle x \rangle_{ab} \Big]$$

for

$$\langle x \rangle_{ab} \equiv \int x \, \psi_a(x) \psi_b^*(x) \, dx$$

$$\langle x_1 x_2 \rangle = \langle x \rangle_a \langle x \rangle_b \pm |\langle x \rangle_{ab}|^2$$

$$\Delta^2 = \underbrace{\langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b} \mp 2 |\langle x \rangle_{ab}|^2$$

$$= \Delta_d^2 \mp 2 |\langle x \rangle_{ab}|^2$$

Since $|\langle x\rangle_{ab}|^2 > 0$

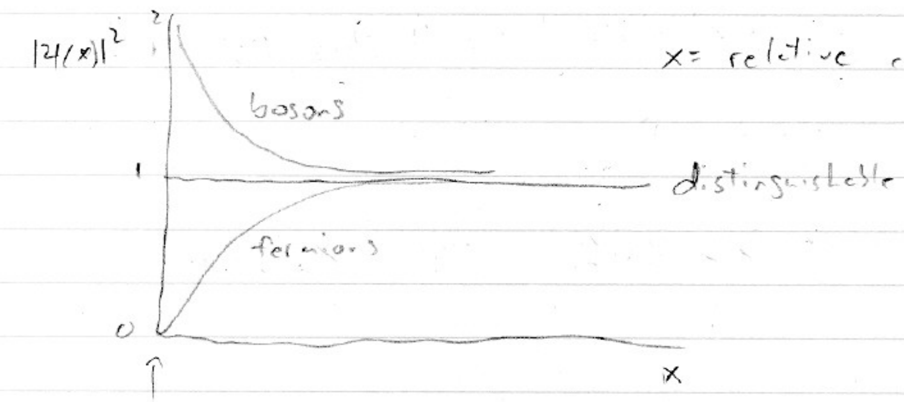
$\Delta^2 < \Delta_d^2$ for bosons

$\Delta^2 > \Delta_d^2$ for fermions

Bosons pulled together by exchange effect
Fermions pushed apart

Called exchange force
Not a force in normal sense
but acts a lot like one

Hard to draw a picture, unfortunately
- related to earlier factor of 2



x = relative coordinate

particles at same location
= in same position eigenstate

distance $\sim \frac{1}{k}$
if
$\psi(r) \sim e^{ikx}$