Lecture 25

Last time, introduced scattering

Describe by $D(\theta) = \frac{d\sigma}{d\Omega} = \frac{(\#\text{ of particles scat into } d\Omega/\text{unit time})}{d\Omega \cdot (\#\text{ of incident particles}/(\text{unit time} \cdot \text{area}))}$

Quantum approach, want $\psi \to A\left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right]$ for large r

with $D(\theta) = |f(\theta)|^2$

One way to get $f(\theta)$: Partial wave analysis

Idea: Can write incident wave e^{ikz} as sum of spherical harmonics Y_ℓ^m

Since scattering potential is symmetric, ℓ & m conserved

So incident $Y_\ell^m \to e^{i\delta} Y_\ell^m$

Scattering information contained in phase shifts

Develop this

Start by focussing on scattered wave $f(\theta)\frac{e^{ikr}}{r}$

We know general solution to Schr eqn written

$$\psi(r,\theta,\phi) = R(r)\,Y_\ell^m(\theta,\phi)$$

where $u(r) = rR(r)$ satisfies

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u = Eu$$

Look at how this behaves for large r

Assume $V(r)$ has finite range.

Then eventually get $\frac{d^2u}{dr^2} = -k^2 u$

$k^2 = \frac{2mE}{\hbar^2}$

$$u(r) = \underset{\text{outgoing}}{C e^{ikr}} + \underset{\text{incoming}}{D e^{-ikr}}$$

We want outgoing wave only (for scattered wave)

so $D = 0$

Thus $R(r) \sim \frac{e^{ikr}}{r}$ as claimed

But lets say r is not quite that big.

Allow $r \gg$ range of $V(r)$, so $V(r) \to 0$
but keep ang. momentum term;

$$\frac{d^2u}{dr^2} - \frac{\ell(\ell+1)}{r^2} u = -k^2 u$$

General solution $u(r) = A r j_\ell(kr) + B r n_\ell(kr)$
spherical Bessel functions

where $j_\ell(r) \sim \sin$
$n_\ell \sim \cos$

But we want to isolate term $\sim e^{ikr}$

Use spherical Hankel functions

$h_\ell^{(1)}(x) = j_\ell(x) + i n_\ell(x) \rightarrow \frac{e^{ix}}{x}$ for large x

$h_\ell^{(2)}(x) = j_\ell(x) - i n_\ell(x) \rightarrow \frac{e^{-ix}}{x}$ for large x

Since these are linear combos of j's & n's, they are valid solutions

Tabulated in book, pg 401

Clearly, we want $h_\ell^{(1)}$ solutions: $R(r) = A h_\ell^{(1)}(kr)$

Put together, complete ψ is

$$\psi(r,\theta,\phi) = A\left\{ e^{ikz} + \sum_{\ell,m} C_{\ell m}\, h_\ell^{(1)}(kr)\, Y_\ell^m(\theta,\phi) \right\} \quad (\text{large } r)$$

We can also eliminate ϕ dependence, on grounds of symmetry:
- incident wave is cylindrically symmetric
- $V(r)$ is cylindrically symmetric
- So scattered wave must be too

So only need terms with $m=0$ in sum

Then use $Y_\ell^0(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$

P_ℓ = Legendre polynomials, page 138

Conventional to define $C_{\ell 0} = i^{\ell+1} k \sqrt{4\pi(2\ell+1)}\; \underset{\substack{\uparrow \\ \text{new coeff.}}}{a_\ell}$

Then $\psi(r,\theta) = A\left[e^{ikz} + k \sum_\ell i^{\ell+1} (2\ell+1)\, a_\ell\, h_\ell(r)\, P_\ell(\cos\theta) \right]$

Call each term in sum a "partial wave"

- a_ℓ is amplitude of ℓ^{th} partial wave = scattering length

Now for very large r, $h_\ell^{(1)}(kr) \to (-i)^{\ell+1} \frac{e^{ikr}}{kr}$

So $\psi \to A\left[e^{ikz} + \sum_\ell (2\ell+1) a_\ell P_\ell(\cos\theta) \frac{e^{ikr}}{r}\right]$

Thus we have $\boxed{f(\theta) = \sum_\ell (2\ell+1) a_\ell P_\ell(\cos\theta)}$

Just need to find a_ℓ's

Can also write $D(\theta) = |f(\theta)|^2 = \sum_\ell \sum_{\ell'} (2\ell+1)(2\ell'+1) a_\ell^* a_{\ell'} P_\ell(\cos\theta) P_{\ell'}(\cos\theta)$

not so nice.

But $\sigma = \int D(\theta)\, d\Omega = 4\pi \sum_\ell (2\ell+1) |a_\ell|^2$

Call $4\pi (2\ell+1) |a_\ell|^2$ = ℓ-wave cross section

Last piece: also need to write e^{ikz} as spherical waves.

Find $e^{ikz} = \sum_\ell i^\ell (2\ell+1) j_\ell(kr) P_\ell(\cos\theta)$

So total wave is

$$\psi(r,\theta) = A \sum_{\ell=0}^{\infty} i^\ell (2\ell+1)\left[j_\ell(kr) + ik a_\ell h_\ell^{(1)}(kr)\right] P_\ell(\cos\theta)$$

at large r.

See how to get a_ℓ's next time.